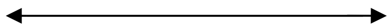
STRAND 1: Lines and angles

Lines

A point can be thought of as an idea of a location. It has neither breadth nor width represented by a dot (.) and denoted by a capital letter. Continuous set of points trace a path which extends in both directions unendingly representing a line. Lines are simply made up of continuous points. If the continuous points are made in a constant direction whether opposite or forward, then straight lines are formed. Straight lines therefore have two unending ends. The direction of the points with respect to the location gives the name of the line as Horizontal, Vertical, Diagonal, etc

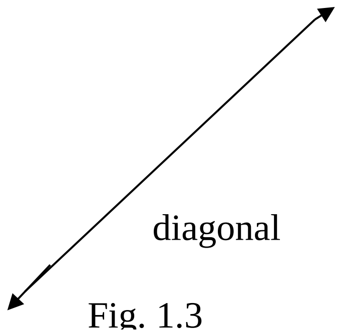


Horizontal

Fig 1. 1

vertical

Fig. 1.2



diagonal

Fig. 1.3

Rays are special straight lines with one ending point and one unending end like below:

Rays

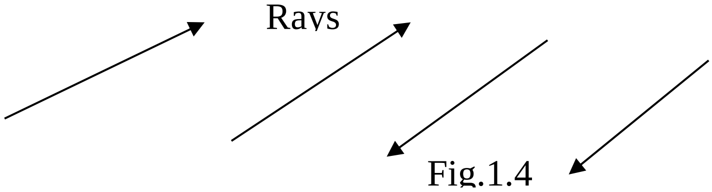


Fig.1.4

A set of rays is called a beam. A real object that can be seen to show a set of rays is the touch light (*try to view it at home*).

When a line has two end points, it is called a line segment. Below are line segments:

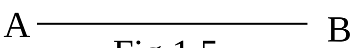


Fig.1.5

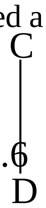


Fig.1.6

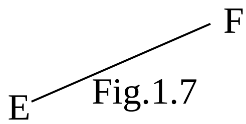


Fig.1.7

Angles

Angles are formed when two lines meet at a common point or when a ray turns or rotates around a fixed point. When you turn a door or window on their hinges, the hinge that the door rotates on is the fixed line or axis of rotation. The amount the door or window moves or turns is the angle it turns through. An angle measures a change in direction. The change of direction from $\overrightarrow{OA}$ to $\overrightarrow{OB}$ may be achieved in two ways:

a) By an anticlockwise rotation of the ray $\overrightarrow{OA}$ into the position $\overrightarrow{OB}$
b) By a clockwise rotation.

Either of these rotations defines an angle between $\overrightarrow{OA}$ and $\overrightarrow{OB}$.

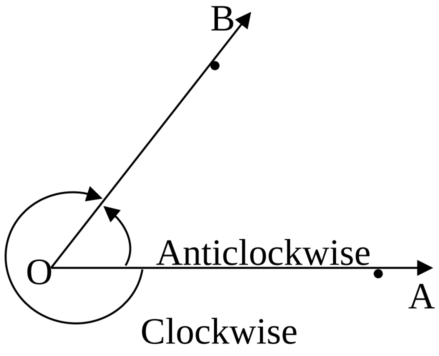


Fig. 1.8

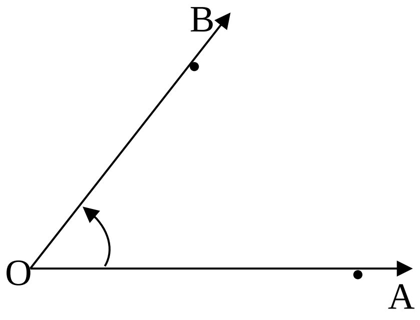


Fig. 1.9

The angle in Fig.1.8 may be named in three different ways:

a) Angle AOB

b) $A\hat{O}B$

c) $\angle AOB$

Types of angles

i. **Right angle** (square angle): A full turn is made up of 360^0. Therefore a quarter turn has $\frac{360^0}{4}$ or 90^0. The illustration is as below.

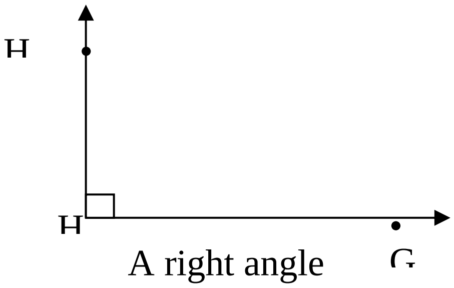


Fig. 1.10

ii. An **Acute angle**: An angle which is less than 90^0 is called an acute angle. It is illustrated in Fig. 1.4

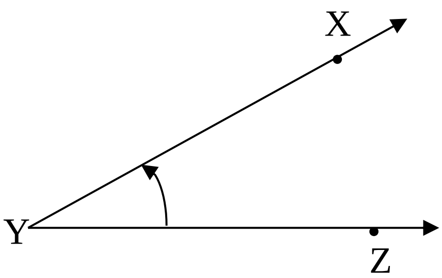


Fig. 1.11

iii. **Obtuse angle**: Obtuse angles are angles that are greater than 90^0 but less than 180^0, as in Fig. 1.5

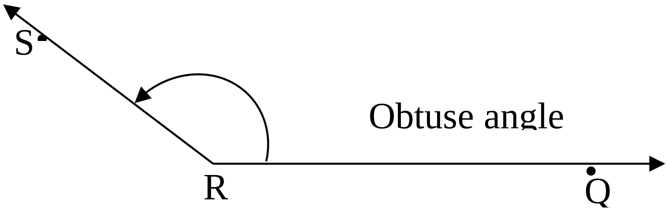


Fig. 1.12

iv. **Reflex angle**: This is an angle that is greater than 180^0 but less than 360^0, as in Fig. 1.6

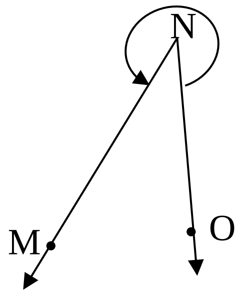


Fig. 1.13

v. **Straight angle**: A straight angle is formed in a half turn as in Fig. 1.7

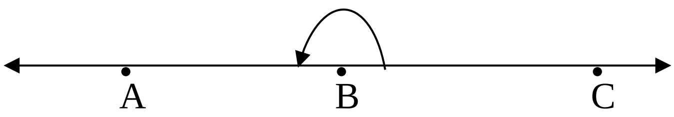


Fig. 1.14

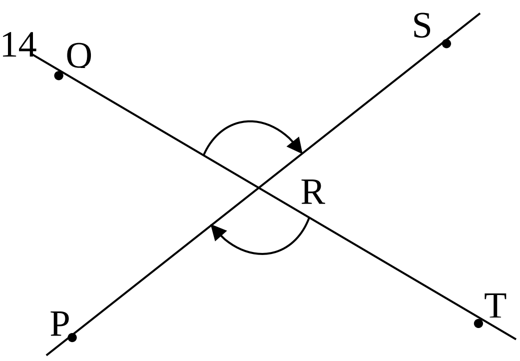


Fig. 1.15

Notes

Vertically Opposite angles are equal. In Fig. 1. 8, the line ST and PS intersect at the point R. A pair of angles such as $\angle QRS$ and $\angle TRP$ are called **vertically opposite** angles and they are equal.

Two angles with a common vertex and a common side are **adjacent** angles.

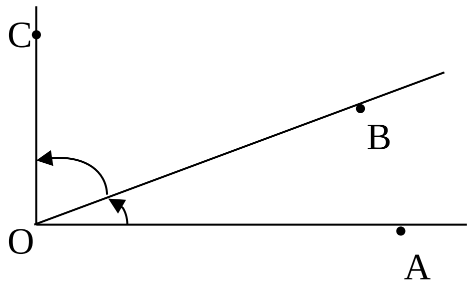


Fig. 1.16

Any two angles that sum up to 180^0 are **supplementary angles**.

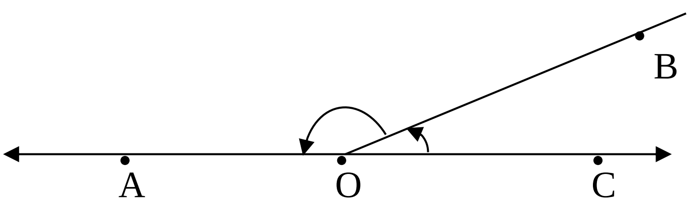


Fig. 1.17

Perpendicular lines

Two lines that intersect to form four right angles are said to be perpendicular

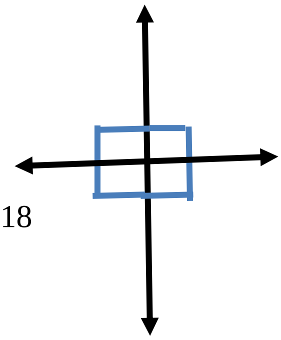


Fig. 1.18

Parallel lines and Transversals

Two or more lines in a plane which never meet no matter how far they are extended in a plane are said to be parallel. The distance between parallel lines are the same throughout the length of their travel. The line that crosses two or more parallel lines are called transversals.

Given two or more parallel lines, if a transversal is drawn across them, the following angles are formed.

i. Alternate angles,
ii. Corresponding angles
iii. Co-interior angles

A) Draw two parallel line AB and CD. Draw a line EF across AB and CD to meet AB and CD at M and N respectively.
   1. Measure $\angle AMN \text{ and } \angle DNM$
   2. Measure $\angle BMN \text{ and } \angle CNM$
   3. What is your observation of the two results?

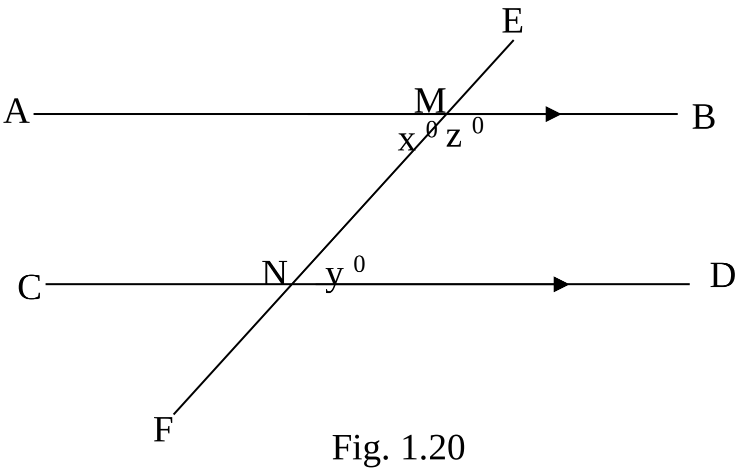


Fig. 1.20

The line EF in Fig. 1.20 is called a transversal. We note that $\angle AMN \text{ and } \angle DNM$ are alternate angles. Also from our measurement, $\angle AMN = \angle DNM$. Similarly, the alternate angles $\angle BMN \text{ and } \angle CNM$ are also equal.

In conclusion, **alternate angles between any two parallel lines and a given transversal are equal.**

i. A measurement of $\angle BME \text{ and } \angle DNM$, $\angle NMB \text{ and } \angle FND$ give same results. $\angle BME \text{ and } \angle DNM$ are corresponding angles, and so also are $\angle NMB \text{ and } \angle FND$.

In conclusion therefore, **corresponding angles formed by any two parallel lines and a given transversal are equal.**

ii. Co-interior angles as in the case of y^0 and z^0 in Fig. 1.20 are supplementary, that is y^0 and z^0 add up to 180^0. Again in summary, **interior opposite angles** between any two parallel lines and a given transversal are **supplementary.**

Example 1.2

AB and CD are parallel lines in Fig.1.21, EF and GH are transversals. Find x, y, z, p, q, and r giving reasons for your answers.

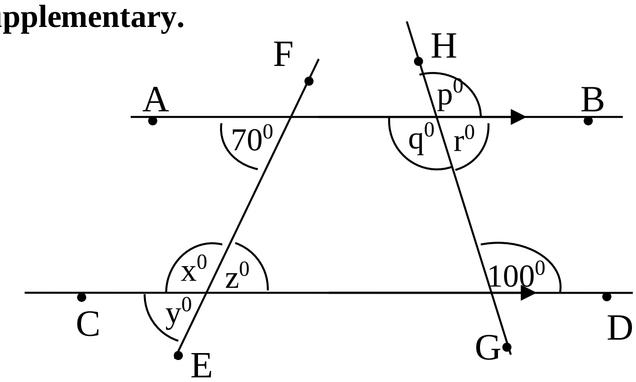


Fig. 1.21

Solution

$r = 80^0$, interior and opposite or co-interior

$y = 70^0$, *y* alternate *z* and its on a straight line with *x*.

$z = 70^0$, either angles on a straight line or alternating angles.

$x = 110^0$, interior and opposite or co-interior angles are supplementary.

$p = q = 100^0$, *p* is on a straight line with *r*, *p* alternates with *q*, also alternates with 100^0 etc.

Example 1.3

Find the values of the angle marked *x*

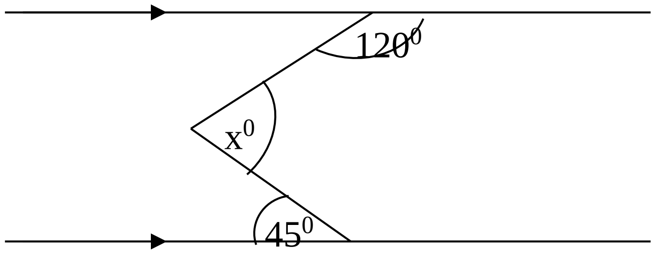
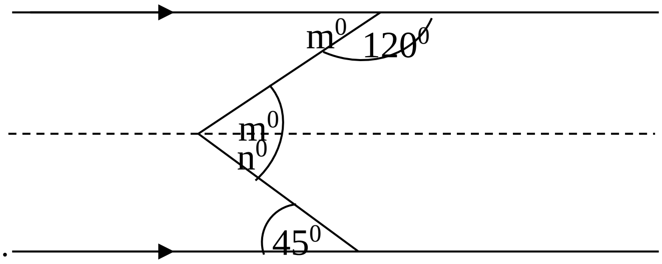


Fig. 1.22

Solution



$m = 60^0$, straight line with 120^0,

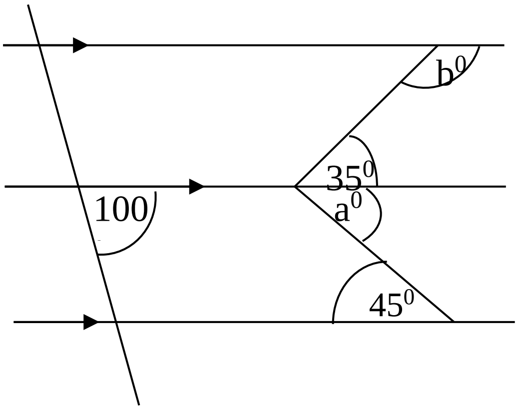
$n = 45^0$, alternating

$\therefore x = m + n = 60^0 + 45^0 = 105^0$.

Fig. 1.23

Example 1.4

Find the values of the angles indicated by letters in the following diagrams.



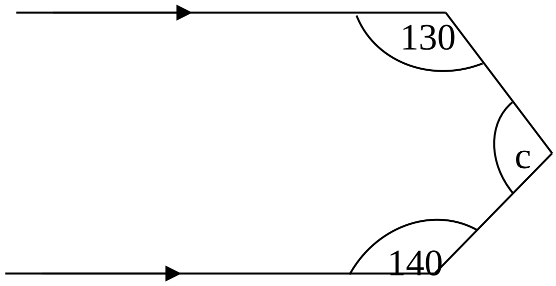


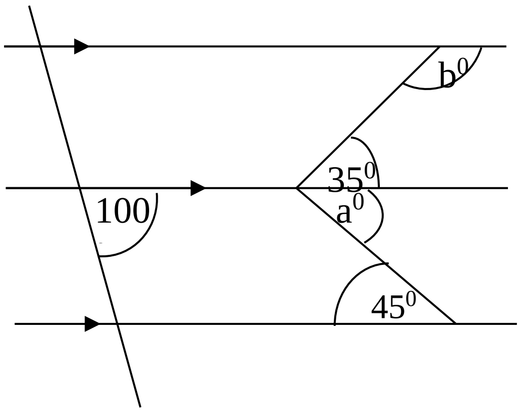
Fig. 1.25

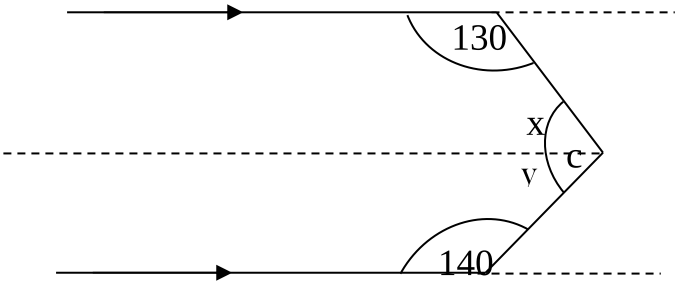
Solution

Fig. 1.24

b = 145 angles on a straight line

a = 45^0 alternating angles





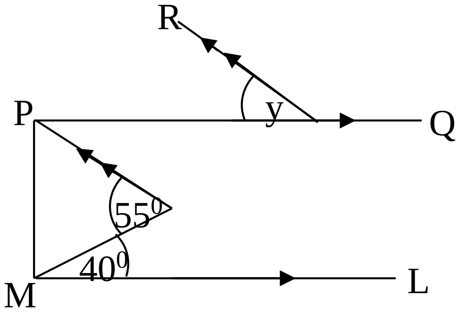
x = 50^0 alternating angles

y = 40^0 alternating angles

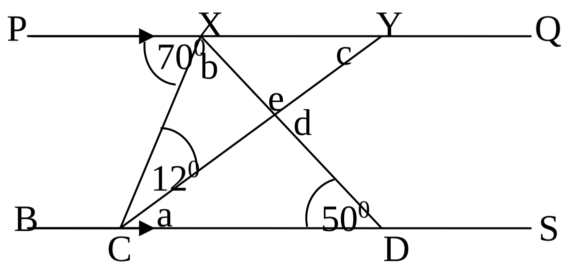
c = x + y = 50 + 40 = 90^0.

Find the size of the angles marked with a letter in the different diagrams below.

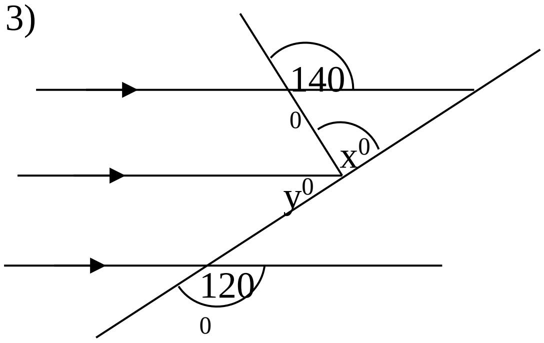
1)



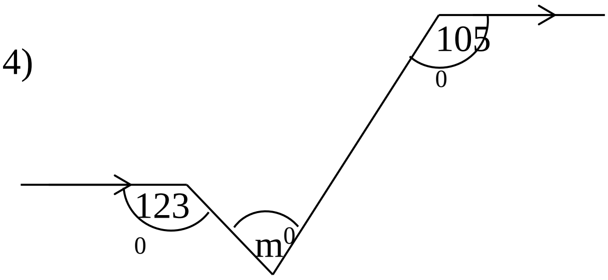
2)



3)



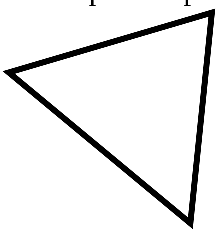
4)



Polygons

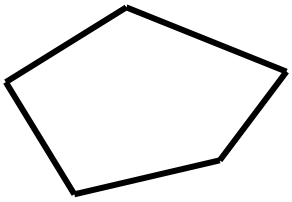
Polygon is a plane figure bounded by line segments. They are special plane figures of the family of geometric shapes. The straight line segments which form a polygon are called sides of the polygon and the point where two lines meet are the vertices. For instance, triangle is a polygon with 3 sides, a quadrilateral is a polygon with 4 sides, pentagon is a polygon of 5sides, etc.

Below are examples of polygons;

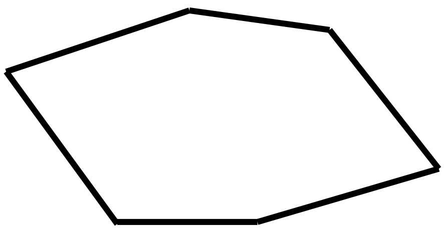


Triangle
Fig 1.1

Quadrilateral
Fig 1.2



Pentagon
Fig 1.3



Hexagon
Fig 1.4

Triangles

A triangle is a closed figure formed by three line segments. We shall now consider the angle properties of a triangle. There are various types of triangle depending on the angles or the lengths of the sides of the triangle.

Types of triangles

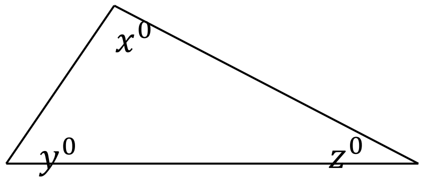


Fig. 1.26

1) **Acute angled triangle:** This is a triangle with each of the three angles being less than 90^0.

2) **Right angled triangle**: A triangle with one of its angles equal to 900. It could be a right-angled isosceles or a right angled scalene.

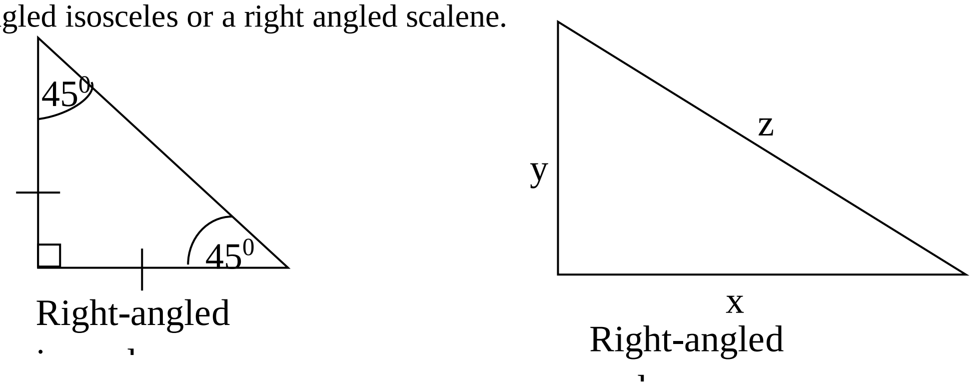


Fig. 1.27

3) **Isosceles triangle**: A triangle with two equal sides and two equal angles. It has one line of symmetry.

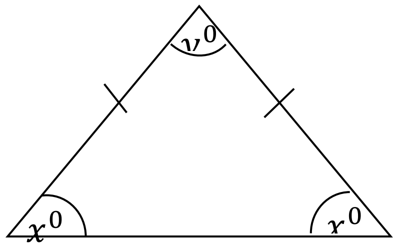


Fig. 1.28

4) **Equilateral triangle:** A triangle with three equal sides and three equal angles. It has three lines of symmetry.

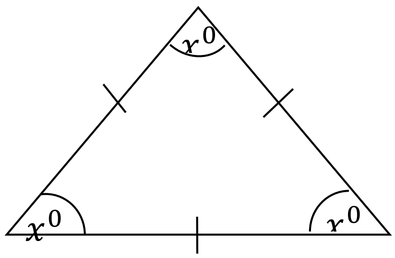


Fig. 1.29

5) **Obtuse-angled triangle**: A triangle with one side being greater than 90^0 but less than 180^0.

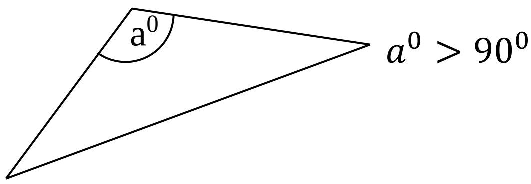


Fig. 1.30

6) Scalene triangle: A triangle with no two sides and no two angles being equal. Scalene has no axis of symmetry.

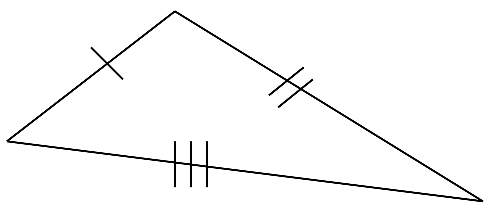


Fig. 1.31

Angle properties of triangles.

1) Angles on a straight line sum up to 180^0.

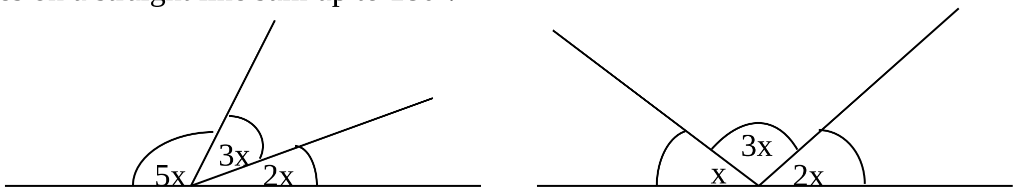


Fig. 1.32

$5x + 3x + 2x = 180^0$ and $x + 3x + 2x = 180^0$ respectively.

2) The sum of the interior angles of a triangle is 180^0.

$75 + 45 + x = 180$
$x = 180 - 120$
$x = 60$.

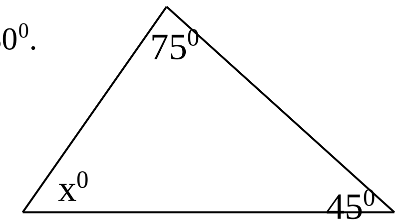


Fig. 1.33

3) The exterior angle of a triangle is equal to the sum of the two interior opposite angles.

Example 1.1
Use the interior angles property of triangles to find the values of *x*, *y* and *z.*

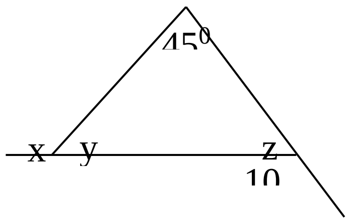


Fig. 1.34

Solution
$x = z + 45^0$, $z = 80^0$, angles on a straight line.
$\therefore x = 80 + 45 = 125$ and similarly
$y + 45 = 100$, $y = 100 - 45 = 55$
$\therefore$ x = 125^0, y = 55^0 and z = 80^0.

4) Angles in equilateral triangle are equal and each is 60^0.

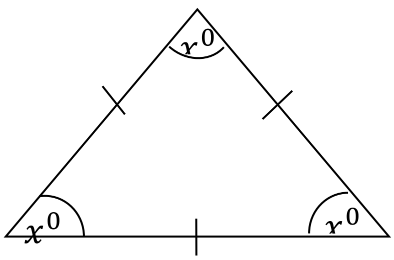
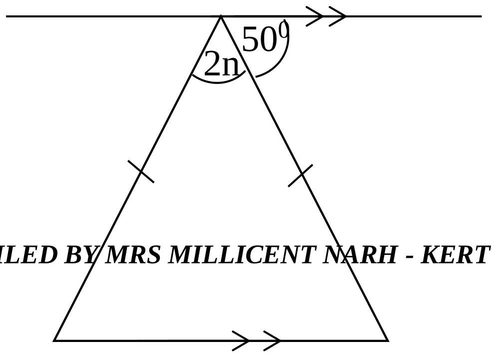


Fig. 1.35

5)



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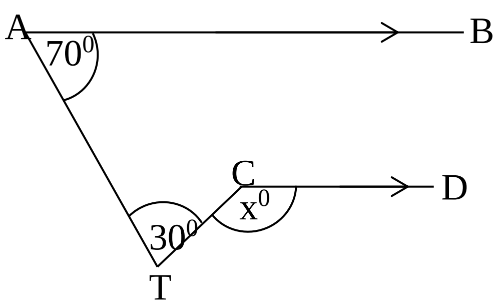

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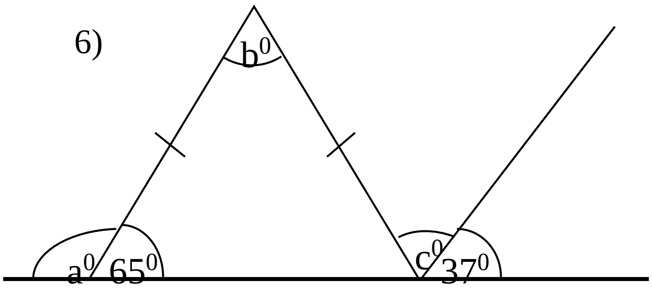

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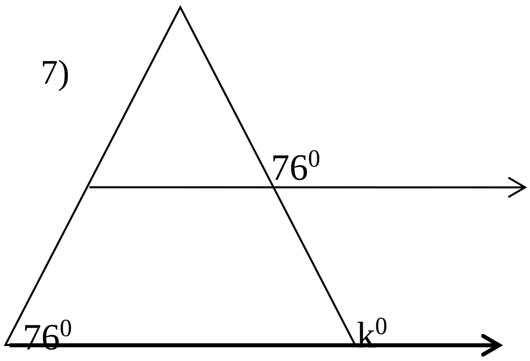

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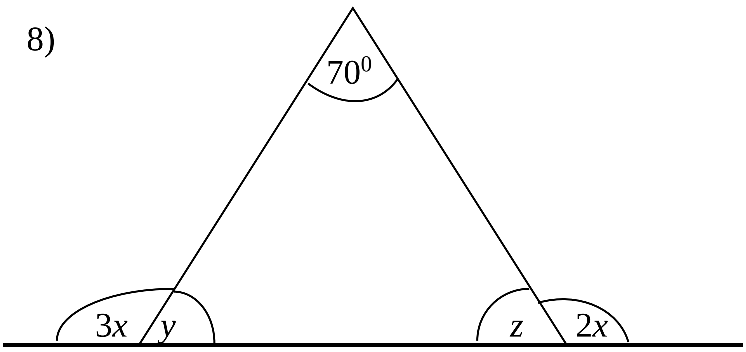

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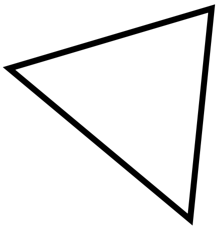
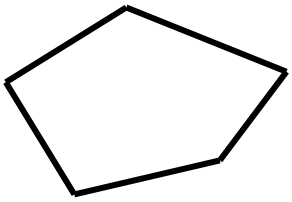
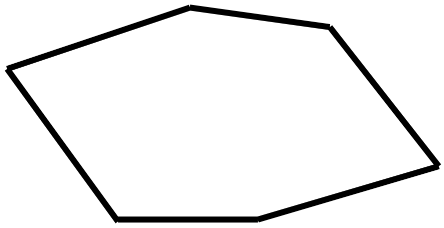


Triangle
Fig 1.1

Quadrilateral
Fig 1.2



Pentagon
Fig 1.3



Hexagon
Fig 1.4

Sum of Angles in a Triangle

You must go through different activities to help you find out that the sum of the interior angles of a triangle is 180^o

To discover that the sum of interior angles of a triangle is 180^o using paper cut outs

- Draw a reasonable triangle on a card
- Indicate the three angles and label them sey 'a', 'b' and 'c'
- Use scissors to cut out the angles and arrange them side by side.
- You will realize that the angles cut out can be arranged on a line
- Since the angle of a straight line is 180^o the sum of the interior angles of a triangle is 180^o

Making a paper model of the sum of interior angles of a triangle

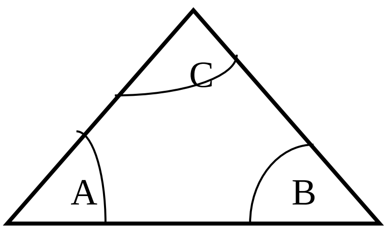


Fig 1.11

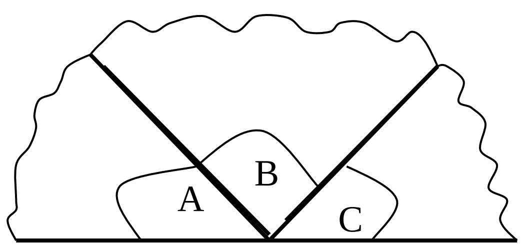


Fig 1.12

To discover that the sum of interior angles of a triangle is 180^o using protractor

Since we are able to draw triangles, let us engage in an activity of drawing and measuring the interior angles of triangles using the protractor. Record the outcome on a table for analyses as below;

student	Angle A	Angle B	Angle C	Total
1	60	61	59	180
2	72	85	23	180
3	90	45	45	180
4	90	60	30	180

Table 1.1

When you study the table, you realize that the sum of interior angles of a triangle is 180^o.

Exterior Angle Properties

Draw any triangle on a card and extend each of the three sides to form three exterior angles. As you can already measure angles, measure the exterior angles and find the sum. You will realize that, the sum of exterior angles of a triangle is 360^o

Study the diagram below

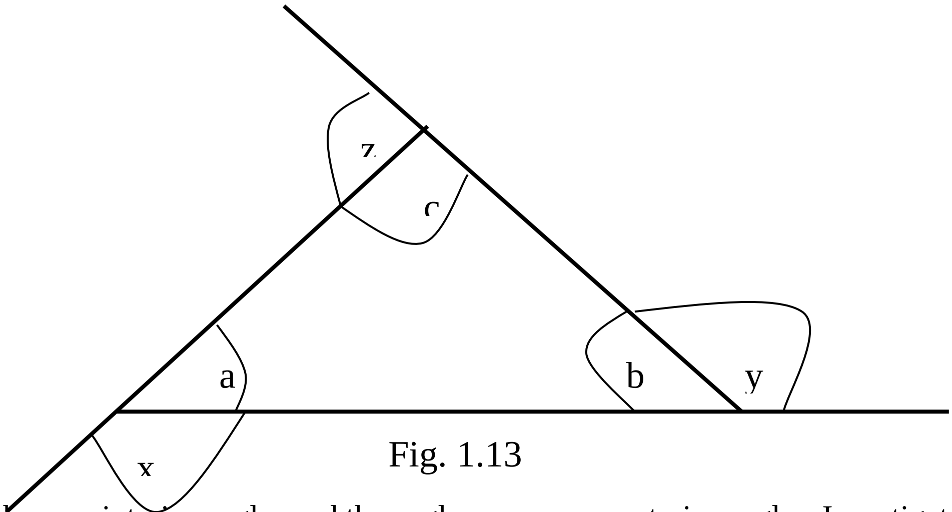


Fig. 1.13

The angle a, b, c are interior angles and the angles x, y, z, are exterior angles. Investigate these six angles, compare and make an inference. You will notice that:

(i) angle y = a + c (ii) x = b + c (iii) z = a + b

Example 1

1) Find x in the figure below;

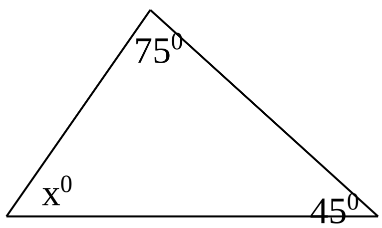


Fig. 1.14

Solution

The sum of the interior angles of a triangle is 180^0.

$75 + 45 + x = 180$

$x = 180 – 120$

$x = 60$.

Example 2

Use the interior angles property of triangles to find the values of *x*, *y* and *z*.

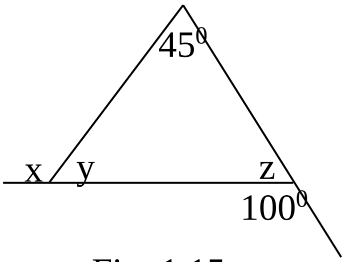


Fig. 1.15

Quadrilateral

Fig 1.18

Solution

The exterior angle of a triangle is equal to the sum of the two interior opposite angles.

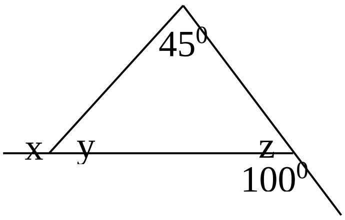


Fig. 1.16

$x = z + 45^0$, $z = 80^0$, angles on a straight line.

$\therefore x = 80 + 45 = 125$ and similarly

$y + 45 = 100$, $y = 100 - 45 = 55$

$\therefore$ x = 125^0, y = 55^0 and z = 80^0.

Sum of Interior and Exterior Angles of other Polygons

The interior angles and exterior angles lies on a straight line and are supplementary. Draw polygons of sides 3, 4, 5, 6 then ask them to draw diagonals from one vertex to all the others such a way as to split the polygons into triangles as below;

Sum of interior angles of polygons

Name of polygon	No. of sides	No. of triangles drawn from only one vertex	Sum of interior angles
Triangle	3	1	$1 \times 180 = 180^0$
Quadrilateral	4	2	$2 \times 180 = 360^0$
Pentagon	5	3	$3 \times 180 = 540^0$
Hexagon	6	4	$4 \times 180 = 720^0$
Heptagon	7	5	$5 \times 180 = 900^0$

Octagon	8	6	$6 \times 180 = 1080^0$
Nonagon	9	7	$7 \times 180 = 1260^0$
Decagon	10	8	$8 \times 180 = 1440^0$
.	.	.	.
.	.	.	.
.	.	.	.
Polygon	n	n – 2	$(n - 2) \times 180$

Table 1.2

This pattern gives the number of triangles always two less than the number of sides. Therefore, the sum of the interior angles of any polygon with n sides as (n – 2) x 180

Example 3

1) Find the sum of the interior angles of a polygon with
a) 7 sides b) 13 sides c) 30 sides

Solution

a) The sum of interior angles of a polygon with n sides is given by $(n - 2) \times 180$,
n = 7 means (7 – 2) x 180 = 900^0
b) n = 13, 11 x 180 = 1980^0
c) n = 30, 28 x 180 = 5040^0

In some special polygons, the sides are congruent and all the angles are also the same. Such polygons are referred to as **regular polygons**. Examples are equilateral triangle and square.

Since the sum of the interior angles of n sides of a polygon is (n – 2) 180^0 and all the angles of the regular polygons are of the same size, each interior angle of the regular ***n*** sided polygon is given as;

$$\frac{(n - 2) \times 180}{n}$$

Example 4

Calculate the size of each interior angle of a regular 10 - sided polygon.

Solution

$$\frac{(n - 2) \times 180}{n} = \frac{8 \times 180}{10} = 8 \times 18 = 144^0$$

Example 5

A regular polygon is such that each interior angle is 150^0. How many sides has the polygon?

Solution

Let the number of sides be ***n***, each angle is 150^0 and the sum = (n – 2)180

$150n = (n - 2) \times 180$

$$150n = 180n - 360$$
$$360 = 180n - 150n$$
$$360 = 30n$$

$$n = \frac{360}{30} = 12^0$$

Example 6

Determine the number of sides of a regular polygon if each interior angle is 125^0

Solution

$$125^0 = \frac{(n-2)\times 180}{n}$$
$$125n = (n-2)\times 180$$
$$125n = 180n - 360$$
$$360 = 180n - 125n$$
$$360 = 55n$$
$$n = \frac{360}{55} = 7$$

Example 7

Three angles of an irregular octagon are 100^0, 120^0, and 140^0. The remaining angles are congruent. Find the size of each of the remaining angles.

Solution

Let each equal angle of the octagon be x hence 5x.

Sum of interior angle of an octagon is = (n - 2) 180

6 x 180 = 1080

100 + 120 + 140 + 5x = 1080

5x=1080 – 360

5x = 720

$x = \frac{720}{5}$

$x = 144^o$

Extend the lesson learnt from sum of interior angles to the sum of exterior angles by extending the sides of the polygons to produce exterior angles on the polygon.

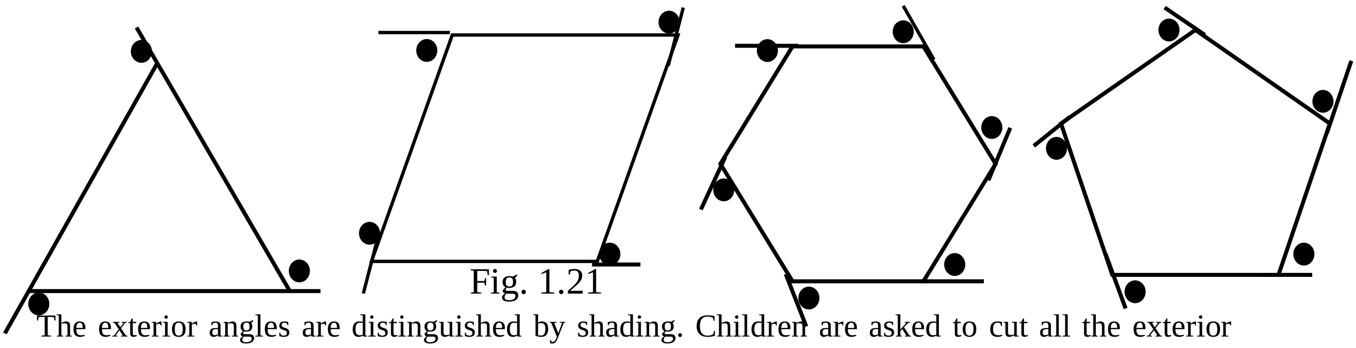
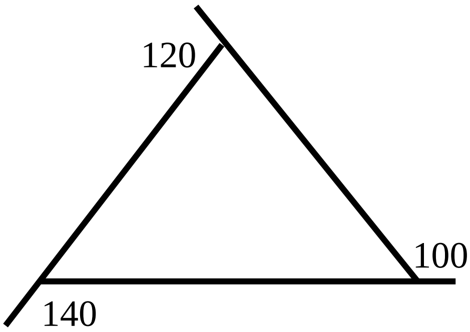


Fig. 1.21

The exterior angles are distinguished by shading. Children are asked to cut all the exterior angles of each polygon. For each polygon on the cut out angles are placed sides by side such that the vertices of the angles involve meet at a common point.

Using the fact that exterior angle plus interior angle = 180, calculate the angles of the exterior angles and find the sum of the exterior angles for each polygon as in figures below;



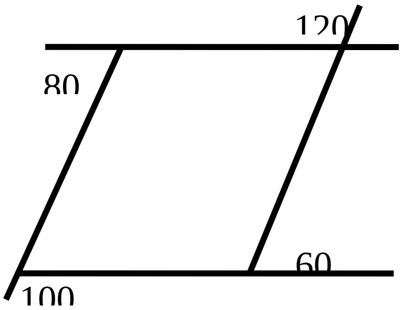
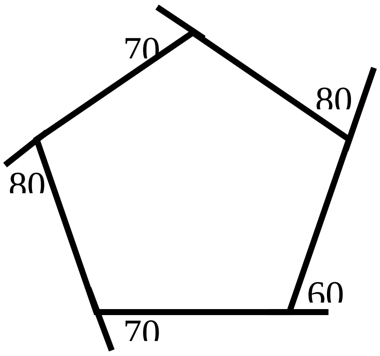


Fig. 1.22



You will realize that no matter the size of the polygon the sum of exterior angles = 360^0

Example 8

A regular polygon has 18 sides,

i. calculate the size of one exterior angle
ii. what is the sum of its interior angles?
iii. From ii, find the size of one interior angle

Solution

The sum of exterior angle of a polygon is 360^o

i) Hence one exterior angle = $\frac{360}{18} = 20^o$

ii) sum of interior angles = (n- 2)180

$(18 – 2)180$

$16 \times 180 = 2880^o$

iii) one interior angle $= \frac{(n-2)180}{n}$

$= \frac{2880}{18} = 160^o$

Try these

1) Find the value of the letters in the figures below.

a)

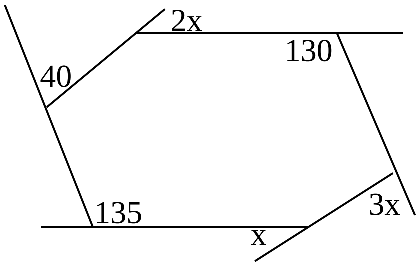


Fig.

b)

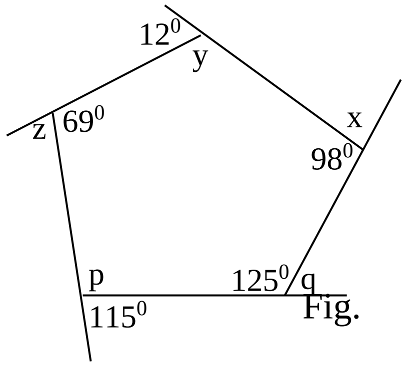
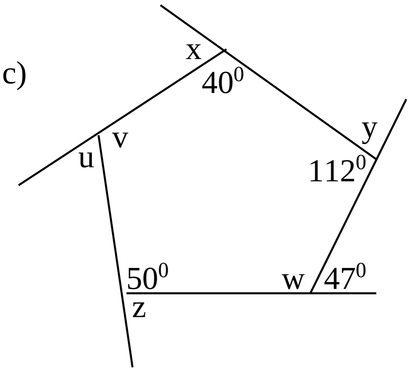


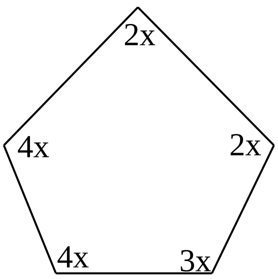
Fig.

c)

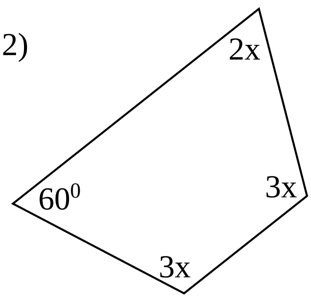


2) Find the sum of the interior angles of a regular polygon with an exterior angle of 72^0
3) The interior angles of a pentagon are in the ratio 2 : 3 : 4 : 4 : 5.
   Find the value of the largest angle.
4) The interior angles of a quadrilateral are y^0, $(2y + 5)^0$, $(y + 15)^0$, and $(3y - 10)^0$. Find the value of y.
5) Determine the number of sides of a regular polygon if the interior angle is 165.6^0
6) The interior angle of a regular polygon is twice its exterior angle. Find the number of sides of the polygon.
7) A regular hexagon is inscribed in a circle of radius 7m. Find the area of the polygon.
8) Find x in the following

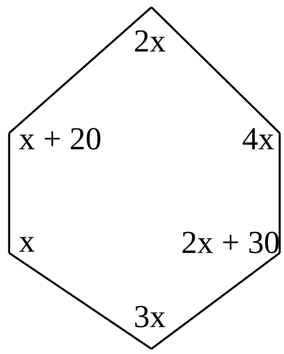
1)



2)



3)



STRAND 2 GEOMETRIC CONSTRUCTION

Introduction

A geometric construction is a drawing of geometric shapes using a compass and a straightedge. When performing a geometric construction, only a compass (with a pencil) and a straightedge are allowed to be used. Most of the ideas we have studied in geometry have been introduced by a Greek mathematician, Euclid, who defined a point as that which has position but no magnitude. Meaning a point has no dimension. For the purpose of construction, a point will be represented by the intersection of two lines. Points are named or referred to by capital letters by Mathematical convention. A line may be straight or curved. A line has length but no width. Again, for the purpose of construction a line will always be understood to be a straight line unless otherwise stated.

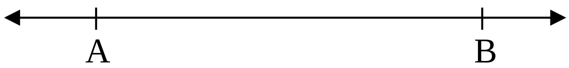


Fig 2.1

The portion of the line from A to B in Fig 3.1 above is called a line segment and is denoted as $\overline{AB}$

There are seven basic geometric constructions.

1. congruent segment
2. segment bisector
3. congruent angle
4. angle bisector
5. a line perpendicular to a given line through a point not on the line.
6. a line perpendicular to a given line through a point on the line.
7. a line parallel to a given line through a point not on the line.

Other geometric shapes or figures, such as right triangles or equilateral triangles, can be constructed using these seven basic constructions.

Constructing a line segment

To construct a line segment $\overline{AB}$ of length 6cm,

a. Draw a line of length greater than 6cm.
b. Using a ruler and a pencil cut the line towards the left end and label that point A.
c. With your pair of compasses, measure 6cm on your ruler.
d. With the point A as origin, draw an arc to cut the line towards the right-hand end.
e. Label the point of intersection of the arc with the line as B
f. Using dividers, check that the line segment $\overline{AB}$ you have constructed is of length 6cm.

Segment Bisector or Perpendicular Bisector

Constructing a perpendicular bisector of a line segment $\overline{AB}$ of length 7cm,

a. Draw your line segment $\overline{AB}$.

b. Open your compass to a measure which is more than half of the length of your segment i.e. 7cm.
c. Put the point of the compass on one end of the segment and construct an arc above and below the segment
d. Without changing the measure of the compass put the point of the compass on the other end of the segment and construct an arc above and below the segment.
e. Draw a segment connecting the two intersections of the arcs.
Note that in the figure AXBY
$|AX|=|AY|=|BX|=|BY|$ since they are all equal radii drawn with the same compass setting.
Therefore AXBY is a rhombus
AB and XY are therefore diagonals of a rhombus.

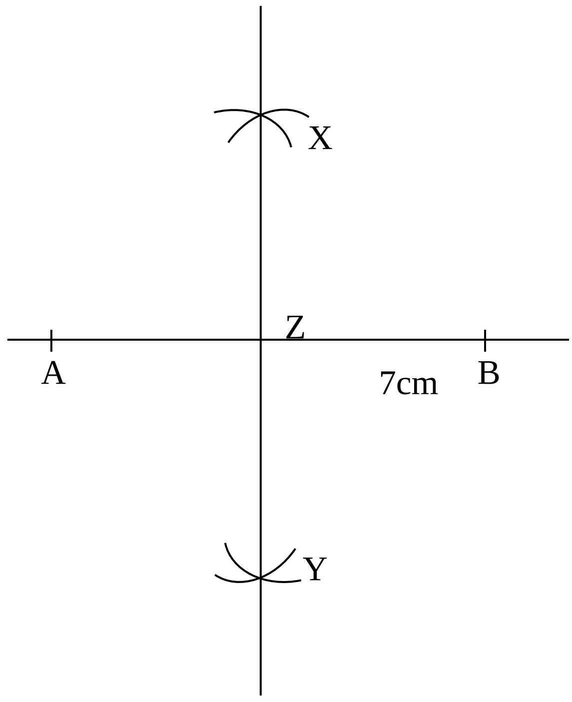


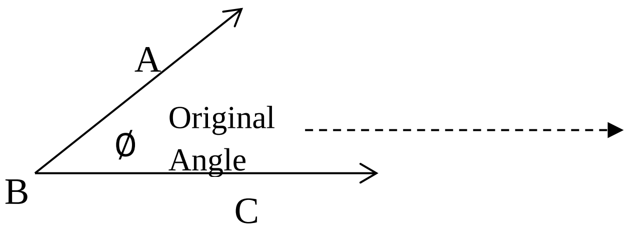
Fig 2.2

Construction and bisection of angles

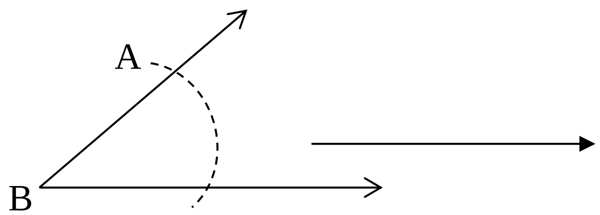
We have learnt earlier that an angle is formed by two intersecting lines.

Congruent angles / Copying angles

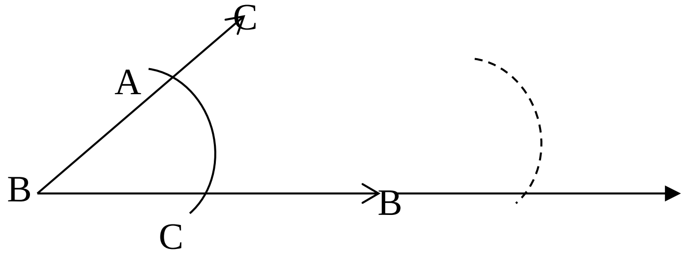
a. Draw a reasonable angle of your choice



b. Construct an arc on the original angle with the point of the compass on the vertex of the angle and the arc crossing both sides of the angle.



c. Without changing the compass, construct the same arc on the ray putting the point of the compass on the end of the ray.



d. Measure the width of the original angle using the compass.

e. Without changing the measure on the compass mark off that width on your ray. Put the point of the compass on the point where the arc crosses the ray and construct an arc crossing your arc.

f. Draw the second side of the angle by connecting the endpoint of the ray (your vertex) with the point where the two arcs intersect.

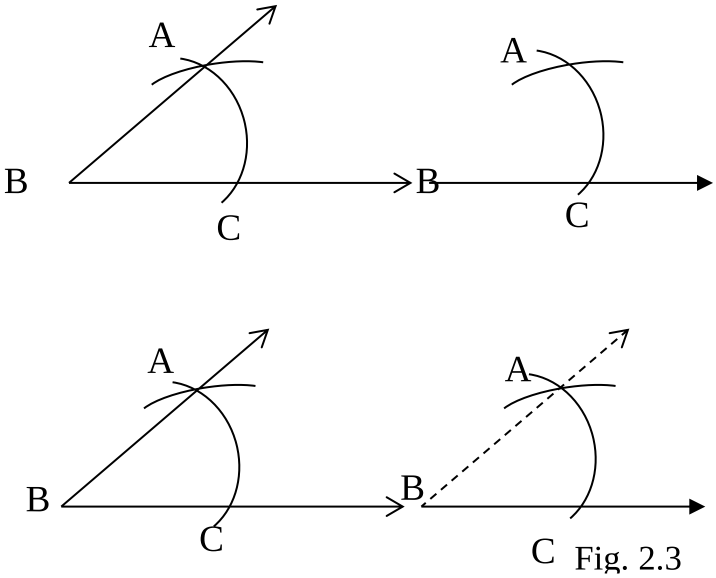


Fig. 2.3

Angle Bisector

Construct an arc crossing both sides of the angle. Put the point of the compass on the vertex of the angle.

a. Construct an arc in the interior of the angle putting the compass on one side of the angle where the arc crosses it.

b. Without changing the compass measure from step 2, put the point if the compass on the other side of the angle where the arc crosses it and draw an arc on the interior of the angle.

c. Draw the angle bisector by connecting the vertex of the angle with the point where the two arcs from steps 2 and 3 cross.

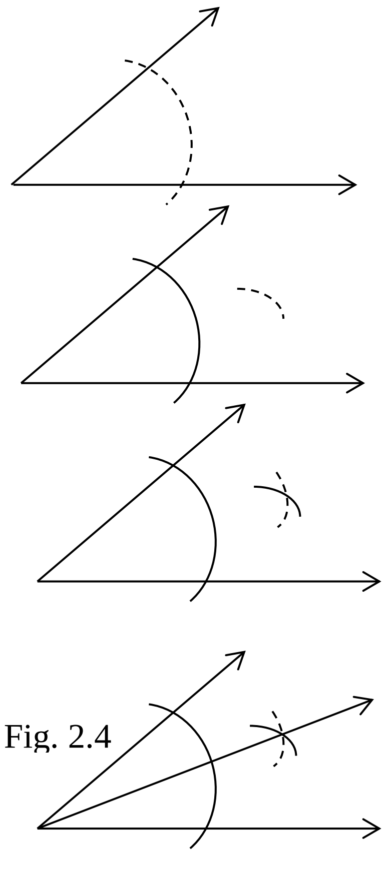


Fig. 2.4

Construction of 120^0 and 60^0.

We will construct 60^0 and 120^0 using the following steps.

a. Draw a line segment say $\overline{AB}$ as shown in the Fig 3.5
b. Locate point X on the $\overline{AB}$
c. With the origin or centre as X with a convenient radius, draw a semi-circle to cut the $\overline{AB}$ at Y
d. Now with Y as the centre or origin and using the same radius in (c), draw an arc to cut the semi-circle at R.
e. From the point X, draw the line $\overline{XC}$ through R.

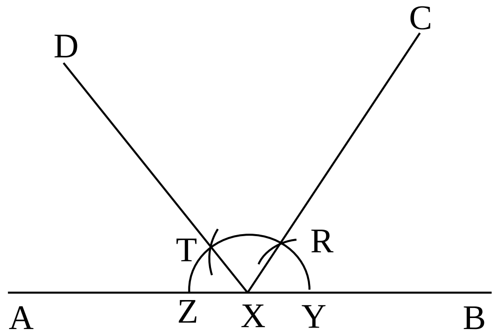


Fig.2.5

f. Measure angle BXC. You are smiling because you had ∠BXC as 60^0.
g. With the origin or centre as R with the same radius in (c), draw another arc to cut the semi-circle at T.
h. From the point X, draw the line $\overline{XD}$ through T.
i. Measure the angle BXD. With accurate geometric construction, angle BXD is expected to be 120^0.

Construction of 30^0 and 15^0.

a. Construct 60^0
b. Bisect 60^0 to get 30^0
c. Bisect 30^0 to get 15^0.

Construction of 135^0, 90^0 and 45^0.

To construct 90^0,

a. Draw a line segment say $\overline{AB}$
b. With the point of your compasses or origin at Z on $\overline{AB}$ and with a convenient radius, draw a semi-circle to cut $\overline{AB}$ at X and Y

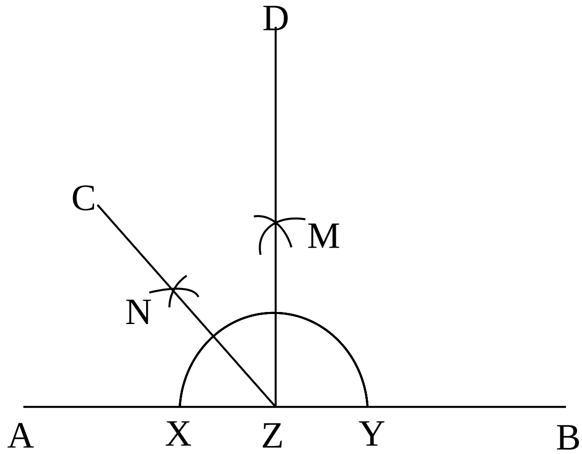
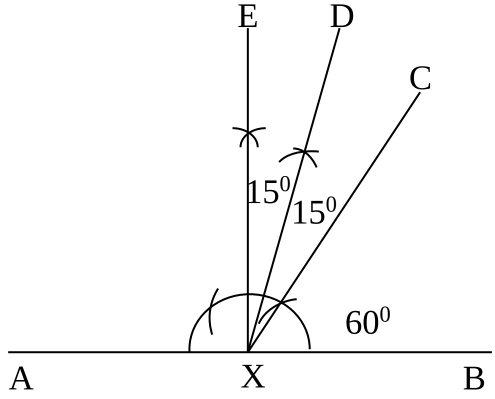


Fig. 2.6

c. With X as the centre or origin, open a convenient radius with your compasses and draw an arc
d. With the same radius, describe another arc using Y as the centre, to intersect the first arc in (c) at M
e. From the point Z, draw the line $\overline{ZD}$ through M. The angle AZD = angle DZB = 90^0.
f. Construct the line $\overline{ZC}$ which is the bisector of angle AZD. Therefore, angle AZC = 45^0 and angle BZC = 135^0.

Constructing 75^0

a. Draw a line segment $\overline{AB}$.
b. Construct 60^0 on $\overline{AB}$.



c. Construct 90^0 again using the same origin say M used in constructing the 60^0.
d. Draw the line $\overline{XD}$ which is a bisector of angle EXC

e. Hopefully, angle BXC = 60^0, angle BXD = 75^0, angle BXE = 90^0.

Bisecting a line segment or a Perpendicular Bisector

a. Open your compass to a measure which is more than half of the length of the line segment $\overline{AB}$.
b. With the origin as A, construct an arc above and below the segment.
c. Without changing the measure of the compass and with the origin or center as B, construct another arc intersecting the first one above and below the segment.
d. Draw a segment connecting the two intersections of the arcs at C and D.

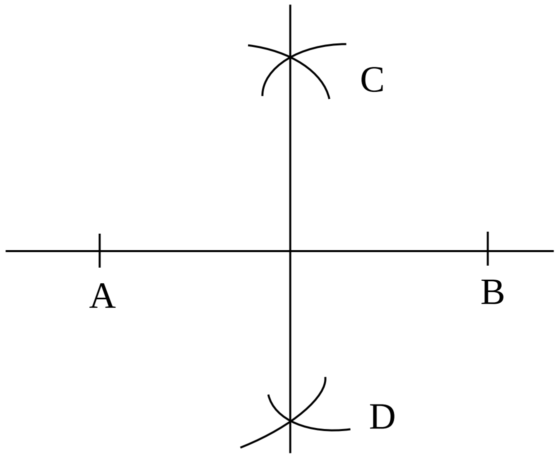


Fig. 2.8

A line perpendicular to a line through a point on the line

a. Put the point of the compass on the point and construct two arcs crossing the line one on each side of the point. <u>Construct a perpendicular bisector of the line segment.</u>
b. Open your compass to a measure which is more than half of the length of your segment.
c. Put the point of the compass on one end of the segment and construct an arc above or below the segment.
d. Without changing the measure of the compass put the point of the compass on the other end of the segment and construct an arc above or below the segment.
e. Draw a segment connecting the intersection of the arcs and the given point.

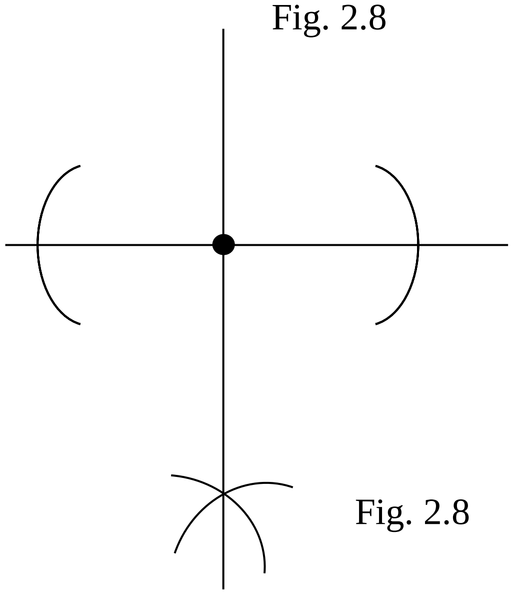


Fig. 2.8

A line perpendicular to a line through a point not on the line

a. Put the point of the compass on the point and construct an arc crossing the line twice once on each side of the point. <u>Construct a perpendicular bisector of the line segment.</u>
b. Open your compass to a measure which is more than half of the length of your segment.
c. Put the point of the compass on one end of the segment and construct an arc above or below the segment.
d. Without changing the measure of the compass put the point of the compass on the other end of the segment and construct an arc above or below the segment.
e. Draw a segment connecting the intersection of the arcs and the given point.

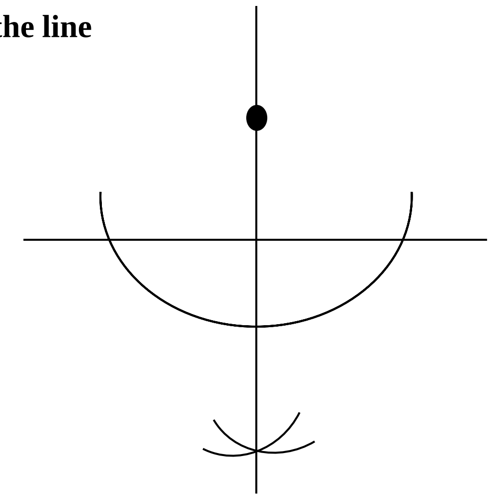


Fig2.9

Construct a line parallel to a given line through a given point

a. Draw a transversal through the point intersecting the line. Construct a congruent angle because if the corresponding angles are congruent then the lines must be parallel.
b. Construct an arc on the angle formed by the transversal and line with the point of the compass on the vertex of the angle and the arc crossing both sides of the angle.

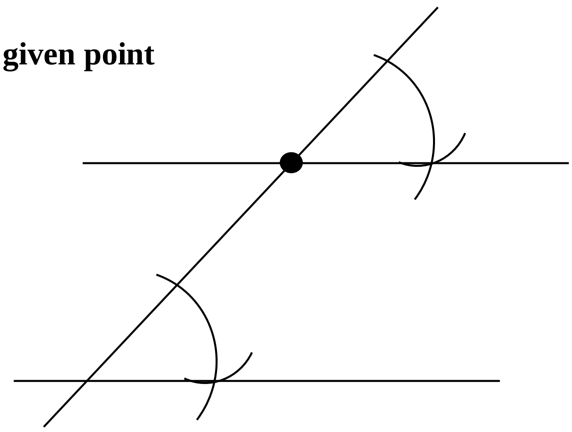


Fig 2.10

c. Without changing the compass, construct the same arc at the point crossing the transversal
d. Measure the width of the angle formed by the transversal and the line using the compass.

e. Without changing the measure on the compass mark off that width at the original point. Put the point of the compass on the point where the arc crosses the transversal and construct an arc crossing your arc.

f. Draw line through the point where the two arcs cross and the given point. Process demonstration

Constructing the lines parallel to a given line at a given distance.

a. Draw the line segment $\overline{AB}$ of length say 7cm.
b. Construct two perpendicular lines at A and B.
c. Open your compass to a radius of 4.5cm and with A as the center or origin, draw an arc to cut the perpendicular drawn through A at P.
d. With B as the centre or origin and with the same radius, (i.e. 4.5cm) draw another arc to cut B at Q.
e. Join P and Q with a straight edge. The line segment $\overline{PQ}$ is parallel to $\overline{AB}$.

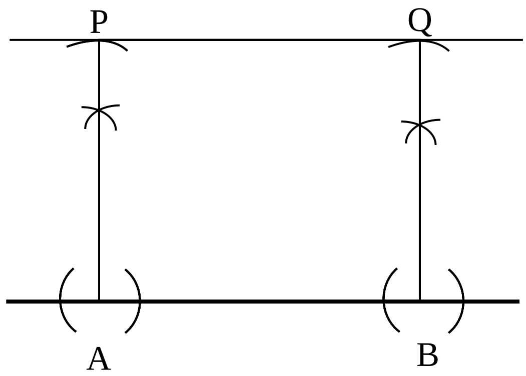


Fig 2.10

Constructing triangles

Triangles can be constructed when the following conditions are given

Three sides (SSS)

Two sides and the included angle (SAS)

Two angles and a side (AAS)

Right angle, hypotenuse and another side (RHS)

<u>Three sides given</u>

Construct triangle PQR such that PQ = 8cm QR = 5cm PR = 6cm

Steps

a. Make a rough sketch of the triangle
b. Construct the line $\overline{PQ}$ = 8 cm
c. With centre P and radius 6cm draw an arc
d. With centre Q and radius 5cm draw another arc to intersect the first arc at R. The point R will then be 5cm from the Q and 6cm P
e. Join P to R and to Q to form the right angle.

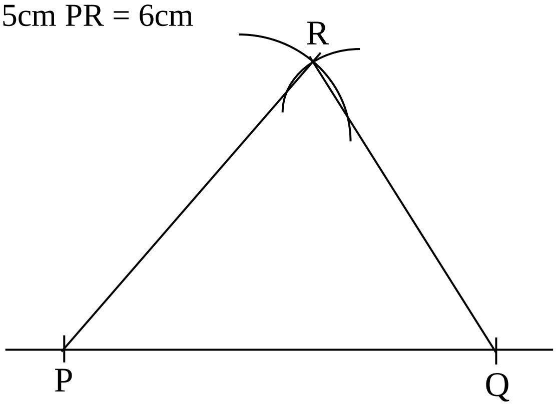


Fig. 2.11

Triangle with two sides and one angle

Construct triangle ABC with $\angle ABC = 45^0, \overline{AB} = 8cm \ and \ \overline{BC} = 6.5cm.$

(i) Make a rough sketch of the triangle

(ii) Draw AB = 8cm,

(iii) Construct angle 45^0 at B (*You first construct the side that contains the angle)*

(iii) With centre B and radius 6.5cm with an arc intersecting the line making 45 with AB at C

(iv) Join AC to complete the triangle

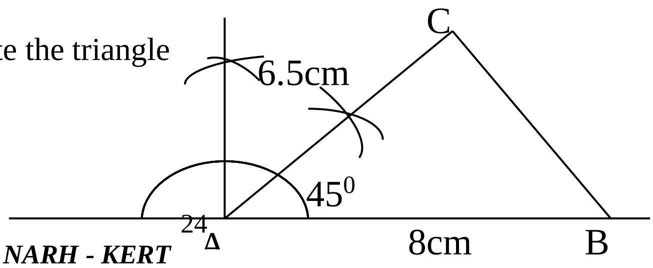


Fig 2.12

Triangle with one side and two angles given

Construct triangle STR such that ST = 7cm, <STR, 45^0, TSR = 30^0

a. Draw the side ST = 7cm
b. Construct angle 45^0 at T
c. Construct angle 60^0 at S
d. Let the two lines meet at R to describe the require triangle

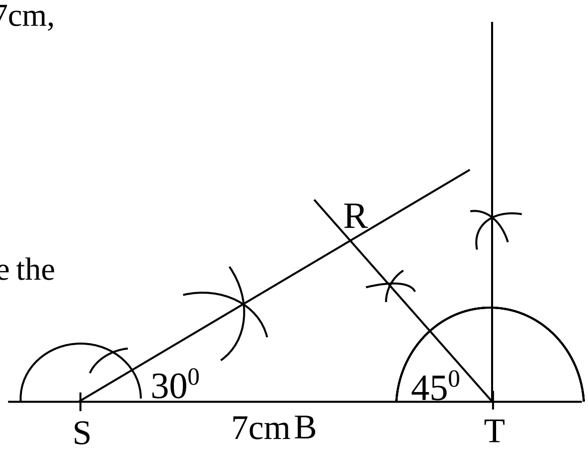
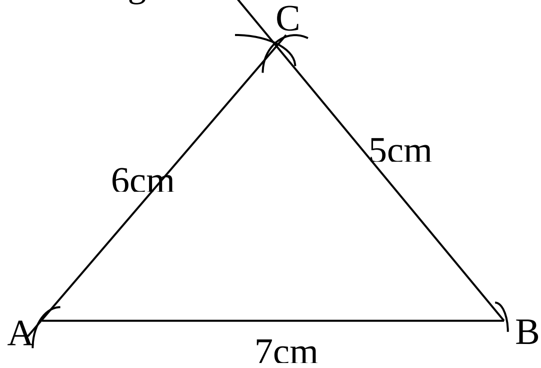


Fig. 2.13

Triangle with three sides and given

a. Draw the side AB = 7cm
b. Measure 6cm with the compass and standing at A strike an arc
c. Measure 7cm with the compass and standing at B strike an arc to cross the first arc
d. Name the intersection C
e. Join the points A and C, also B and C



Try the following

1) Using a ruler and a pair of compasses only, construct triangle ABC such that $|AB| = 8cm, |BC| = 7cm \text{ and } \angle ABC = 105^0$. Measure $|AC|$.
2) Using a ruler and a pair of compasses only, construct ΔABC such that $|AB| = 7cm, \angle ABC = 75^0 \text{ and } \angle BAC = 60^0$. Measure $|AC| \text{ and } |BC|$.
3) Using a ruler and a pair of compasses only, construct ΔABC such that $|AB| = 10.4cm$, $\angle BAC = 90^0 \text{ and } \angle ABC = 30^0$. Measure $|AC| \text{ and } |BC|$.
4) Using a ruler and a pair of compasses only, construct ΔABC such that $|AB| = 5.5cm$, $|AC| = 6cm \text{ and } \angle BAC = 135^0$. Measure $|BC|$.

Construction of Quadrilaterals

A quadrilateral has four sides and four angles. To construct a quadrilateral you need the three conditions as given in the construction of a triangle, plus a fourth condition. For the four vertices of the quadrilateral, three of them can be determined as we construct a triangle. The fourth condition will fix the fourth vertex of the quadrilateral.

Construct the rectangle PQRS, with $|PQ| = 6cm \text{ and } |QR| = 5cm$

Steps

1. Construct the line segment $\overline{PQ}$ of length 6cm
2. Construct 90^0 at P
3. Construct another 90^0 at Q
4. With P as the origin or center and a 5cm radius, draw an arc to cut the perpendicular at P at the point S
5. Also with Q as the centre or origin and with same radius draw an arc to cut the perpendicular at Q at the point R
6. Join R to S as shown in Fig. 3.14
   PQRS is the required rectangle with dimension 5cm by 6cm.

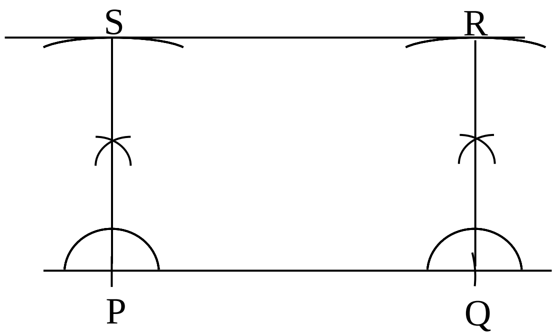


Fig.2.14

With the aid of a pair of compasses and a ruler, construct a parallelogram ABCD such that $|AB| = 7cm,\ \angle BAC = 60^0,\ and\ \angle ABC = 75^0$. Measure $|BC|$

Steps

1. Construct the line segment $\overline{AB} = 7cm$
2. Construct and angle of 60^0 at A
3. Construct an angle of 75^0 at B
4. Extend the lines that make angles of 60^0 at A and 75^0 at B to meet at C.
5. With a radius of 7cm and with your origin at C, draw an arc on the left of C
6. With same radius or magnitude as BC, and with your origin as A, draw another arc to intersect the first arc in (5) at D.
   The quadrilateral ABCD is as shown in fig. 3.15

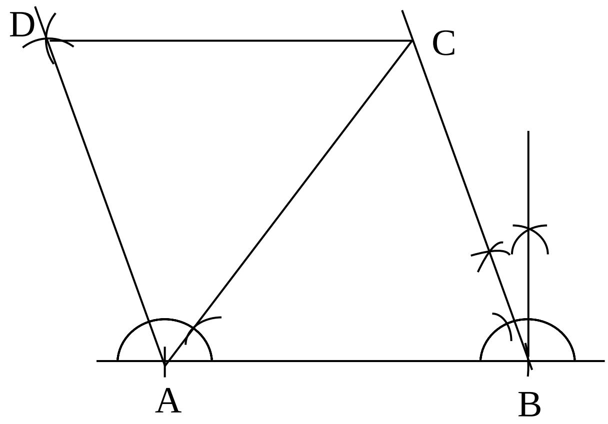


Fig. 2.15

Construct a quadrilateral ABCD with AB = 4cm, BC = 6cm, $\angle ABC = 60^0$
CD = 6.5cm and AD = 7.0cm

Steps;

Make rough sketch

1. Construct the line AB = 4cm
2. Construct angle 60^0 at B. With 6cm radius and B as the origin locate C on the 60^0 line from B.

3. With centre at C and radius 6.5cm draw an arc
4. With centre A and radius 7.0 draw an arc to meet the first arc at D
5. Join DC and DA to complete the quadrilateral

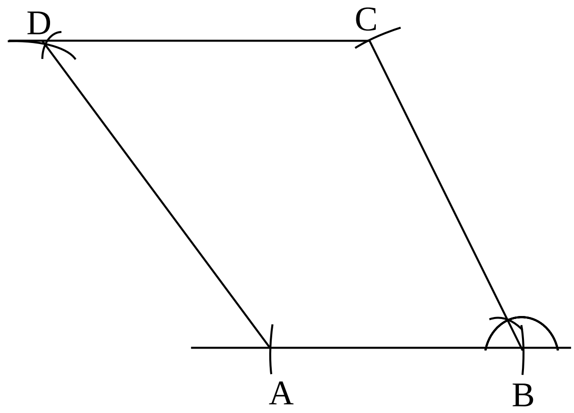


Fig. 2.16

Construction of a Regular Hexagon

A hexagon is polygonal with six sides. Construct a regular hexagon of side 3cm

Steps

Draw a circle of radius 3cm

Using the same radius of the circle place the compass point anywhere on the circumference and draw an arc on the circumference

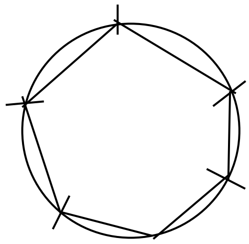


Fig. 2.17

Place the point of the compasses on at the end of the arc, and draw another arc on the circumference.

Repeat the process until you come to the point where you started

Join the arcs with straight lines as in the diagram in fig. 3.17

Circumcircle and Inscribed Circle

A circle is said to a circumscribe when it passes through the three points of a triangle. A circle is called an inscribed circle when it touches all the sides of a triangle

Circumcircle

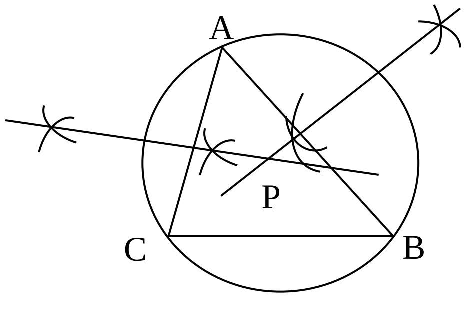
The locus of point P equidistant from three points A, B and C is the point of intersection of the perpendicular bisectors of the sides of the triangle ABC.

A circle drawn through one of the points way A with center P will pass through the other two points B and C of the triangle. The circle drawn is called the circumcircle of triangle ABC and the centre P is called the circumcentre.

To Construct a Circle through The Vertices of A, B and C of the Triangle, Bisect all the three sides of a triangle

Locate where the bisects meet

With this point as a centre draw a circle passing through all the three points

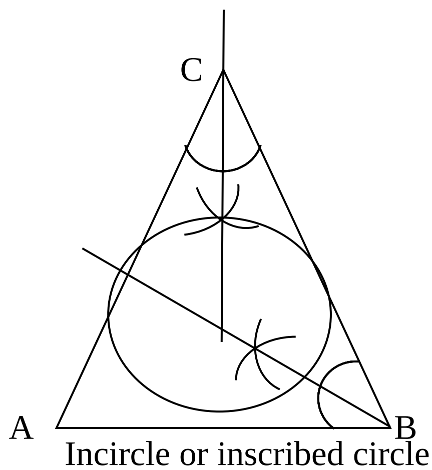


Circumcircle

Fig 2.18

Inscribed or incircle

The points A, B and C are the vertices of a triangle. If the point P is equidistant from the three line segments $|AB|$, $|BC|$ and $|AC|$, then the locus of P is the point of intersection of the bisectors of the angles of the triangle formed by the lines. A circle drawn with P as the centre to touch any of the sides say $|AB|$, will also touch the other sides $|BC|$ and $|AC|$. The circle is called incircle of the triangle ABC and the centre P is called the incentre. To locate the point P which is equidistant from $|AB|$, $|BC|$ and $|AC|$,



Incircle or inscribed circle

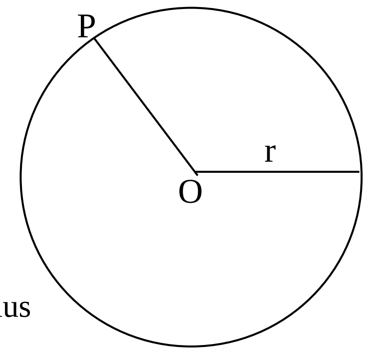
Fig. 2.19

Bisect the three angles of the triangle, find the point where the bisectors meet. Draw a circle from this point to which all the sides of the triangle see Fig 2.19 **Note** that it is not necessary to find all three bisectors. Two bisectors are enough to determine the centre of the circle.

Elementary Ideas of Locus

Locus is the path of points which move in a plane in relation to other points in the plane. At the elementary work, we talk about four different types of locus. These loci are usually derived from our basic constructions

(i) **The Circle:** This is the locus (path) of points which moves in such a way that its distances from a fixed point say 0 is always the same i.e. The circle is the locus which is equidistant from one point.



Note: The fixed point is the centre and the distance is the radius

Fig. 2.20

Example 1

Using a ruler and a pair of compasses only,

a) Construct triangle ABC in which $|AB| = 10cm,\ |BC| = 6cm\ and\ \angle ABC = 45^0$

b) Locate a point D inside the triangle ABC such that D is equidistant from $\overline{AB}\ and\ \overline{AC}$ and 5cm from B.

c) Construct a straight line through D to cut AB at X and AC at Y such that $\overline{AX} = \overline{AY}$

d) Measure $|AY.|$

Solution

a) To construct the triangle,

1. Draw a line segment $\overline{AB}$ of length 10cm
2. Construct 45^0 on B and locate C which is 6cm on the 45^0 line from B.
3. Joint A to B and to C to form the triangle.

b) If D is equidistant from the intersecting straight lines $\overline{AB}\ and\ \overline{AC}$, then D is on the line bisecting $\angle BAC$ and 5cm from B.

c) Construct a perpendicular line at D to cut $\overline{AB}\ at\ X\ and\ \overline{AC}\ at\ Y.$

d) From Fig. 3.20, $|AY| = 6.1cm$.

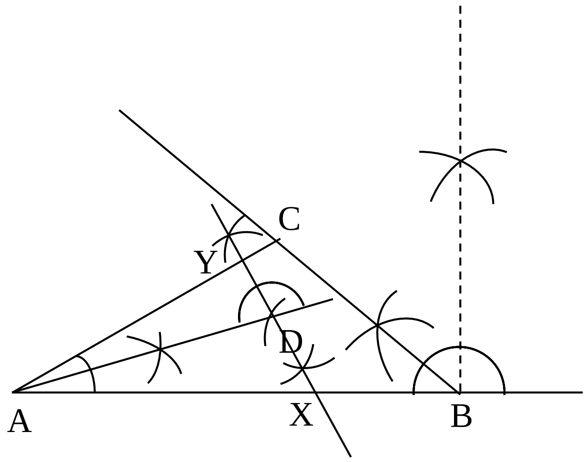


Fig. 2.21

Example 2

Using a pair of compass and a ruler only,

a) Construct ΔPQR such that $|PQ| = 9cm,\ \angle PQR = 75^0,\ and\ \angle QPR = 60^0$

b) Locate a point T, inside ΔPQR such that it is equidistant from $\overline{RQ},\ \overline{RP}\ and\ \overline{PQ}$

c) Construct a circle which touches the three sides of ΔPQR and measure its radius.

Solution

a) Construct ΔPQR by

   1. Constructing the line segment $\overline{PQ} = 9cm.$
   2. Construct 60^0 on P and 75^0 on Q. The intersection of the lines forming the two angles is R.

b) If the point T is equidistant from the three line segments $\overline{RQ},\ \overline{RP}\ and\ \overline{PQ}$, the T is the poi t of intersection of the bisector of the angles of the triangle formed by the lines.

c) From T, draw a perpendicular to PQ and with the perpendicular distance from T as radius, and T as centre, draw a circle which will touch all the three sides of ΔPQR.

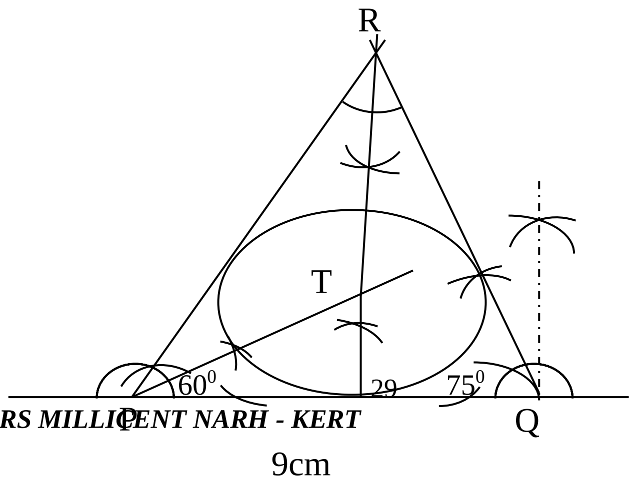


Fig 2.22

The radius of the circle is 3cm.

Example 3

a) Using a ruler and a pair of compass only, construct

1. Triangle ABC, with $|AB| = 7cm, |AC| = 8cm\ and\ \angle BAC = 105^0$.
2. X, the locus of points 6cm from C.
3. Y, the locus of points equidistant from $\overline{AB}\ and\ \overline{BC}$ to cut X in P and R

b) Measure

1. $|BC|$
2. $|PR|$

Solution

a) Construct

1. ΔABC
2. X is a circle with centre C and radius 6cm.
3. Y is a straight line bisecting the angel formed by $\overline{AB}\ and\ \overline{BC}$, which cut x in P and R

b) 1. $|BC|$ = 11.8cm

2. $|PR|$ = 8.5cm

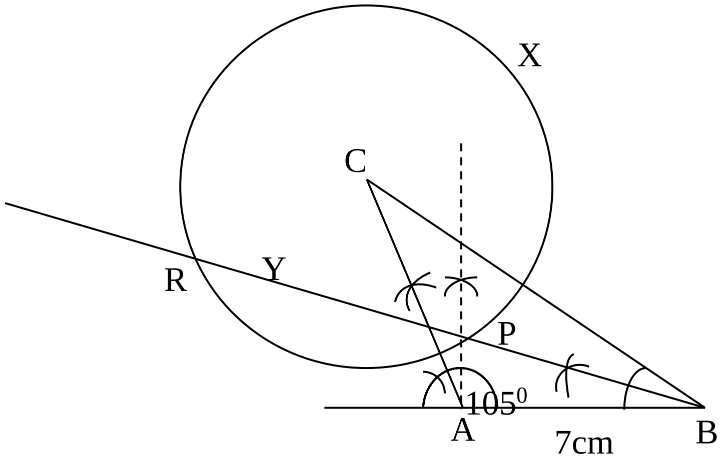


Fig. 2.23

Example 4

Using a ruler and a pair of compasses only

a) Construct

i. Triangle ABC such that $|AB| = 8cm,\ \angle ABC = 30^0\ and\ \angle BAC = 105^0$

ii. The locus l_1 of points equidistant from A and B

iii. The locus l_2 of points equidistant from B and C

b) Locate P, the point of intersection of l1 and l2.

c) Using PC as radius draw a circle

d) Measure

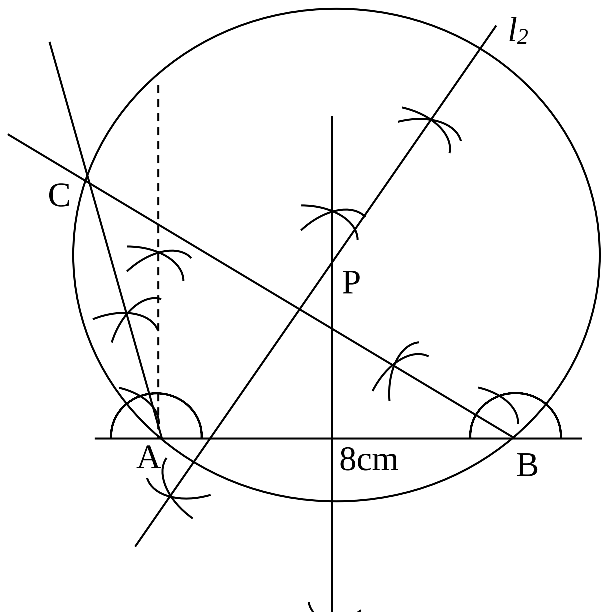
i. $|BC|$

ii. The radius of the circle.

Solution

a) Construct

1. ΔABC
2. The locus l_1 is the bisection of the line segment $\overline{AB}$.
3. The locus l_2 is the bisection of the line segment $\overline{BC}$.



b) P is the point of intersection of l_1 and l_2.
c) The required circle with P as centre and CP as radius is shown in Fig. 3.23
d) 1. $|BC| = 11cm$
   2. The radius of the circle is 5.7cm.

Example 5

a) Using a ruler and pair of compasses only
   1. Construct ΔABC such that $|AB| = 8cm,\ \angle ABC = 60^0\ \ and\ \angle BAC = 75^0$
   2. Locate the point O inside triangle ABC equidistant from A, B and C.
   3. Construct the circle with centre O, which passes through A

b) Measure
   1. $|OA|$
   2. Angle ACD

Solution

a) Construct
1. ΔABC
2. O is equidistant from three points A, B and C. Meaning it is the point of intersection of the perpendicular bisectors of sides of ΔABC.
3. A circle drawn through A, with centre O will pass through the other two points B and C of ΔABC.

b) 1. $|OA| = 5.7cm$
   2. angle ACD = 45^0

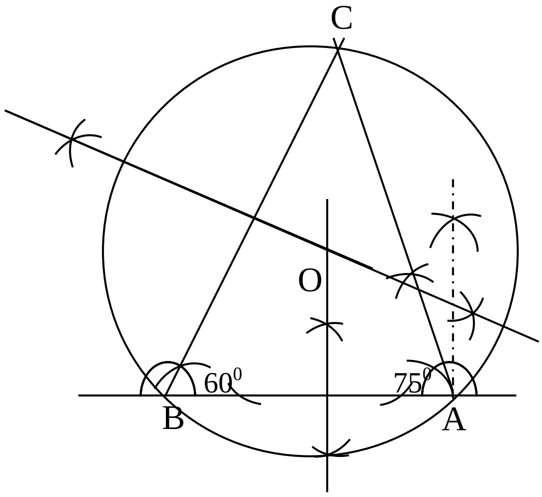


Fig. 2.25

Example 6

a) Using a ruler and a pair of compasses only, construct
   i. ΔGBC with $\angle GBC = 30^0,\ |BC| = 9.5cm\ \ and\ |BG| = 12cm$
   ii. l_1, the locus of points 6cm from C.
   i. l_2, the perpendicular from C to BG.
b) 1. Locate A and D, the intersection of l_1 and BG.
   2. Measure $|AD|\ \ and\ \ \angle ACD$
   3. Calculate, correct to two significant figures, the area of the minor sector ACD.
   [Take $\pi = 3.142$]

Solution

a) Construct
   i. ΔGBC
   ii. The locus l_1
   iii. The locus l_2

b) Locate
   1. A and D
   2. $|AD| = 7.3cm$, $\angle ACD = 76^0$
   3. $r = 6cm \; and \; \theta = 76^0$

Area of a sector = $\frac{\theta}{360} \times \pi r^2$

= $\frac{76}{360} \times 3.142 \times 6^2$

= 23.9cm^2 = 24cm^2 (2 s.f.)

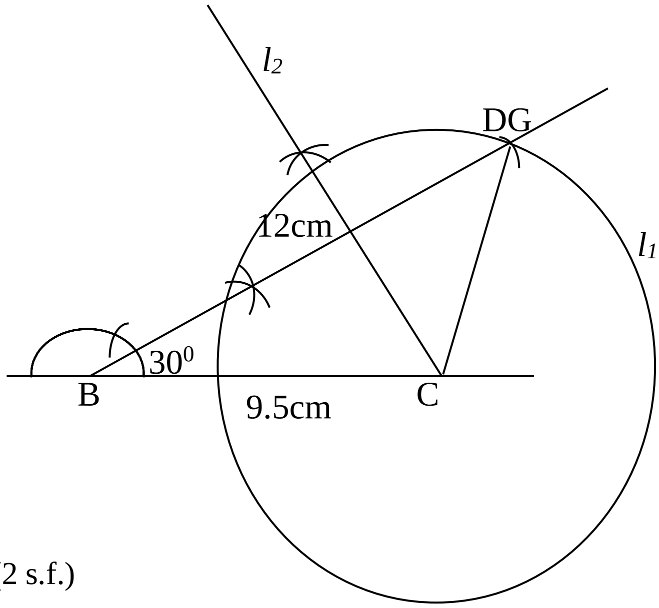


Fig. 2.26

Try these

1. a) Using a ruler and a pair of compass only, construct a quadrilateral PQRS such that $|PQ| = 8cm$, $Q\hat{P}S$ = 105, $\angle PQS = 30^0$, $|PR| = 9cm \; and \; |RS| = |RQ.|$
   b) Measure:
      i. $|RS|$
      ii. $|PS|$
      iii. Angle QRS.
2. Using a ruler and a pair of compasses only, construct triangle ABC in which $\angle BAC = 45^0, |AB| = 7cm \;\; and \;\; |AC| = 9cm$. Locate P, inside the triangle ABC, 5cm from A and equidistant from B and C.
3. a) Using a ruler and a pair of compasses only, construct:
      i. A quadrilateral ABCD, with $|AB| = 8cm,\; |AD| = 6cm,\; |BC| = 10cm,$ $\angle BAD = 60^0 \;\; and \;\; \angle ADC = 135^0$
      ii. The locus l_1, of points equidistant from $|BC|$ and $|CD|$.
      iii. The line l_2, from B perpendicular to l_1.
   b) i. Locate E, the point of intersection between l_1 and l_2
      ii. Measure $|DE|$.
4. Using a ruler and a pair of compasses only,
   a) Construct:
      i. ΔABC with $|AB| = 8cm,\; \angle ABC = 30^0 \; and \; \angle BAC = 120^0$.
      ii. The locus l_1, of points equidistant from A and C
      iii. The locus l_2, of points equidistant from AB and AC
   b) Locate the point of intersection D, of l_2 and BC
   c) i. Construct locus l_3 of points 2.5cm from D
      ii. Locate E and F, the point of intersection of l_1 and l_3 and measure $|EF|$.
5. a) Using a ruler and a pair of compasses only, construct
      i. Triangle ABC in which $|AB| = 7cm,\; |AC| = 8cm \;\; and \;\; \angle BAC = 105^0$.
      ii. l_1, The locus of points 5cm from C
      iii. l_2, the locus of points equidistant from $\overline{AB}$ and $\overline{BC}$ to cut l_1 in P and

b) Measure;

i. |BC|

ii. |PR|

6. Using a ruler and a pair of compasses only,

a) construct triangle PQR such that $|PQ|$ = 10cm, $|QR|$ = 12cm and $\angle PQR = 60^0$

b) Locate a point T, inside ΔPQR such that it is equidistant from $|RQ|$, $|RP|$ and $|PQ|$.

c) Construct a circle which touches the three sides of ΔPQR and measure its radius.

7. A point P is in the same plane as the fixed line segment XY.

a) Construct the locus L of the point P if it moves in the plane such that $\angle XZY = 90^0$ and $|XY| = 10$cm

b) Locate a point Z on the locus L such that $|XZ|$= 7cm. Measure $|YZ|$

8. Using a ruler and a pair of compasses only,

a) Construct:

i. ΔPQR such that $|PQ| = 10cm, \ |QR| = 12cm \ and \ \angle PQR = 60^0$

ii. The locus l_1 of the set of points equidistant from $|RP| \ and \ |RQ|$

iii. The perpendicular l_2 from P to $\overline{QR}$

b) X is the intersection of l_1 and the perpendicular l_2 to $\overline{QR}$.

c) Measure $|RX|$

9. Using a ruler and a pair of compasses only,

i. Construct a parallelogram PQRS with |PQ| $= 10$cm, |PS| $= 8cm \ and \ \angle QPS = 120^0$

ii. Construct the bisectors of $\angle PSR \ and \ \angle QPS$ and mark their point of intersection O:

iii. Construct the line OA from O to meet SR at right angle at the point A

iv. Hence draw a circle to touch the sides PQ, PS and SR. Shade the region which lies within the circle and less than 8cm from Q

10. Using a ruler and a pair of compasses only, construct ΔABC with AB = 6cm, AC = 8cm, and BAC = 75^o.

a) Construct

i. The locus l_1 of points equidistant from AB and AC

ii. The locus l_2 of points equidistant from A and B.

iii. Locus l_3 of points 4.5cm from B.

b) Locate

i. The point of intersection P_1, of l_1 and l_2.

ii. The point of intersection P_2 of l_1 and l_3 inside ΔABC

iii. The point of intersection P_3 of l_2 and l_3 inside the triangle.

c) Measure $\angle P_2P_1P_3$

STRAND 3: BASIC TRIGONOMETRY:

LEARNING, TEACHING AND APPLYING

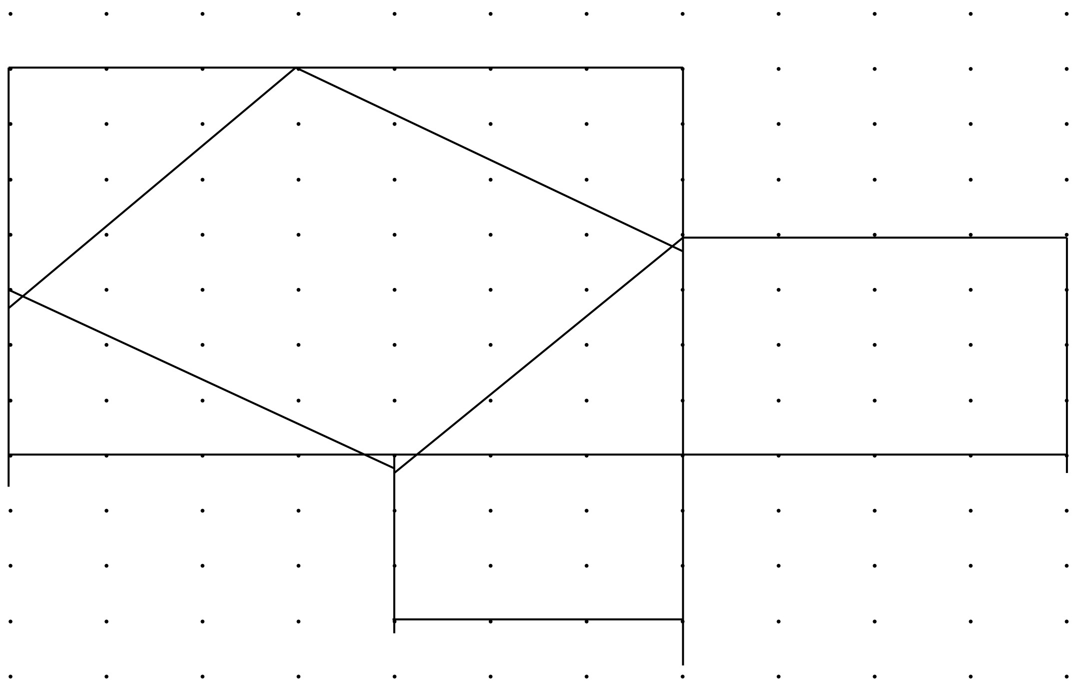
Pythagoras Theorem

Pythagoras theorem states that the sum of the squares of the two shorter sides of a right- angled triangle equals the square of the longer side which is called the hypotenuse. You may go through the following activities to establish the formula.

Activity 1: Using Geoboard

Construct right angled triangle of shorter sides 3 cm and 4 cm on the geoboard.

On each side of this triangle they should construct squares whose sides are equal to the sides of the triangle. The two smaller sides or the shorter sides to be the length 3 + 4 i.e. the side 3 is extended by 3 to form a square of side 7 and of area = 49 squares



Each corner triangles has an area of ½ x 4 x 3 = 6 squares

=> Area of the 4 corner triangles = 4 x 6 = 24 squares

This gives the area of the larger square to be 49 – 24 = 25 squares. Repeat the activity for about three other right angled triangles and put your results on a table like below:

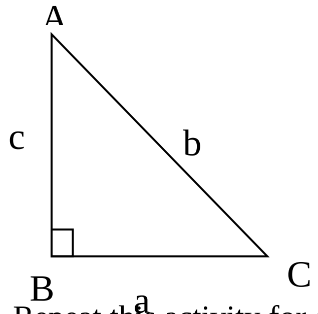
Area of Squares on Shorter		**Sum of squares on shouter side**	**Area of squares of hypotenuse**
4 x 4 = 16	3 x 3 = 9	16 + 9 = 25	25
5 x 5 = 25	2 x 2 = 4	25 + 4 = 29	29
1 x 1 = 1	3 x 3 = 9	1 + 9 = 10	10
2 x 2 = 4	3 x 3 = 9	4 + 9 = 13	13

From the table, we notice that the sum of the areas on the squares of the two shorter sides = the area of the square on the hypothenus

Activity 2: Using Constructions:

Construct a right angled triangle ABC of sides BC = 4cm and AB = 3cm and a right angle at B on a square paper

Join A to C. Measure AC. Square AB, BC, and AC



a	c	b	a^2	c^2	b^2	$a^2 + c^2$
3	4	5	9	16	25	9 + 16 = 25
8	6					
5	12					
8	15					

Repeat this activity for other right -angled triangles of the following sizes.

(i) AB = 2 units BC = 3 units <B = 90^0 (ii) AB = 4 units BC = 2 units <B = 90^0 (iii) AB = 5unit BC = 3 units B = 90^0 (iv) AB = 2 units, BC = 5 unit <B = 90^0.

Right angled-triangle

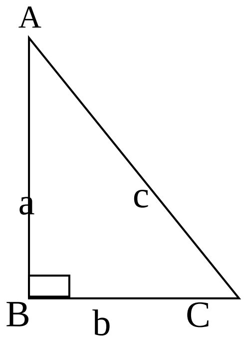


Fig. 3.1a

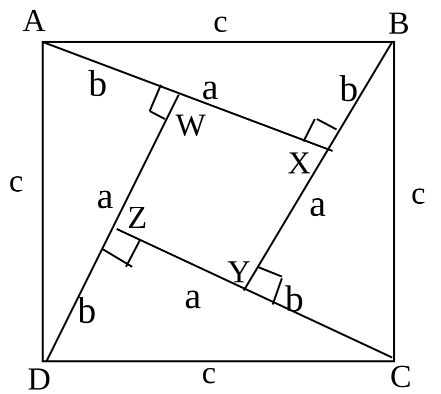


Fig. 3.1b

Cut out four congruent right-angled triangles and rearrange them so that their hypotenuses form the sides of a square as shown in Fig. 4.1 a and b.

From the triangles $|DW| = |CZ| = |BY| = |AX| = a - b$

Area of $ABCD = c \times c = c^2$ but the area of ABCD is made up of a square WXYZ and four right-angled triangle.

Area of $WXYZ = (a - b)^2$

Area of each right angled-triangle = $\frac{1}{2} \times base \times height = \frac{1}{2}ab$

Area of the 4 right angled-triangle = $\frac{1}{2}ab \times 4 = 2ab$.

Area of $ABCD = area\ of\ WXYZ + area\ of\ the\ 4\ right\ angled\ triangles$

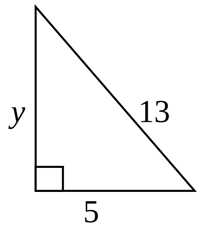
$c^2 = (a - b)^2 + 2ab = a^2 + b^2 - 2ab + 2ab = a^2 + b^2$.

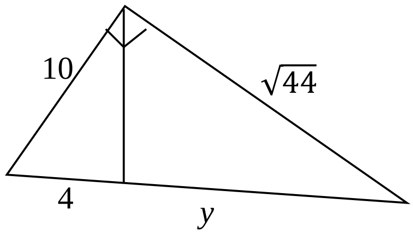
$c^2 = a^2 + b^2$

The implications are that, for any right angled triangle, the square of the hypotenuse is equal to the sum of the squares of the other two sides.

Example 1

Find the length of the *y* in the figures below.





Solution

$13^2 = 5^2 + y^2$

$y^2 = 13^2 - 5^2 = 169 - 25 = 144$

$y = \sqrt{144} = 12$

$y = 12$

b) $(4 + y)^2 = 10^2 + \left(\sqrt{44}\right)^2 = 100 + 44$

$4 + y = \sqrt{144} = 12$

$y = 12 - 4 = 8$

Example 2

The width of a garage is 3.0m, and the roof rises symmetrically to a ridge. The length of one half of the sloping roof is 1.7m. The walls are 2.0m high above ground level. Calculate the distance of the highest point of the roof from the floor of the garage.

Solution

The diagram of the garage is as shown in Fig. 5.2

$|AD|^2 + |BD|^2 = |AB|^2$

$|BD|^2 = |AB|^2 - |AD|^2 = 1.7^2 - 1.5^2 = 2.89 - 2.25$

$|BD| = \sqrt{0.64} = 0.8$

The height of the garage is 2.0m + 0.8m = 2.8m.

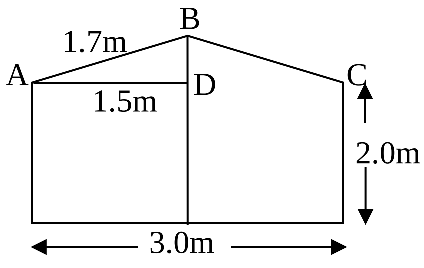


Fig. 3.2

Example 3

The length of the diagonal of a square is (a + b) units. Show that the area of the square is $\frac{1}{2}(a + b)^2$ squared units

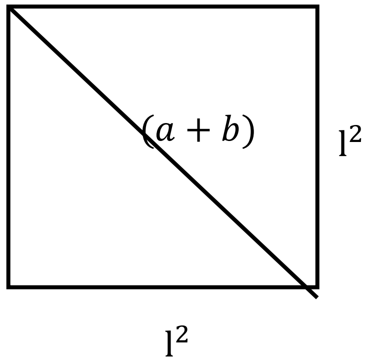
Solution

From Pythagoras theorem,

$l^2 + l^2 = (a + b)^2$

$2l^2 = (a + b)^2$

$l^2 = \frac{1}{2}(a + b)^2$



Try these

1) A boy placed a ladder of length 3m so that one end just reaches the top of a wall. He measured the distance from the foot of the ladder to the wall and found it to be 2.5m. Calculate the height of the wall.
2) The position of three villages X, Y and Z form a right angled triangle. Village Y is 6km directly south of village X and 6km west of the village Z. A road is to be constructed from village X to Z to link the three villages. What is the shortest distance from Y to the road?
3) A solar collector and its stand are in the shape of a right triangle. The collector is 5.00 m long, the upright leg is 2.00 m long, and the base leg is 4.58 m long. Because of inefficiencies in the collector's position, it needs to be lowered by 0.50 m on the upright leg. How long will the new base leg be? Round to the nearest tenth.
4) A builder wants to buy some corrugated aluminum sheets to make a roof of a garage 3.6m wide, as indicated in the diagram. The roof is to slope symmetrically to a ridge 2m above the top of the walls, allowing 50cm overlap at each end. Find the length of sheet he should buy.

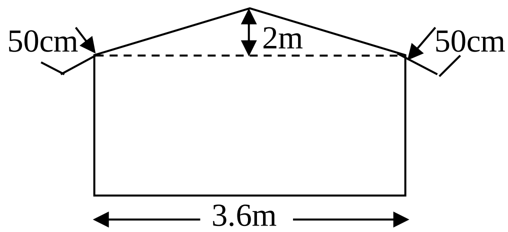


Fig.3.3

5) Find the length and the perimeter of a rectangle whose diagonal is 9 in. and width 6 in.

Trigonometric Functions

Signs of the trigonometrical ratios or functions depend on the quadrant in which the terminal side of the angle lies. The quadrants of the circle are counted in anti-clockwise direction as shown below.

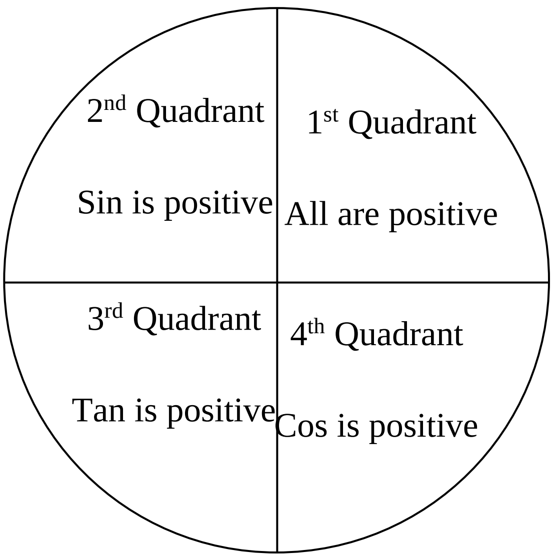


Fig. 3.4

In First quadrant : $x > 0,\ y > 0 \Rightarrow sin\theta = \frac{y}{r} > 0,$ $cos\theta = \frac{x}{r} > 0,\ tan\theta = \frac{y}{x} > 0.$

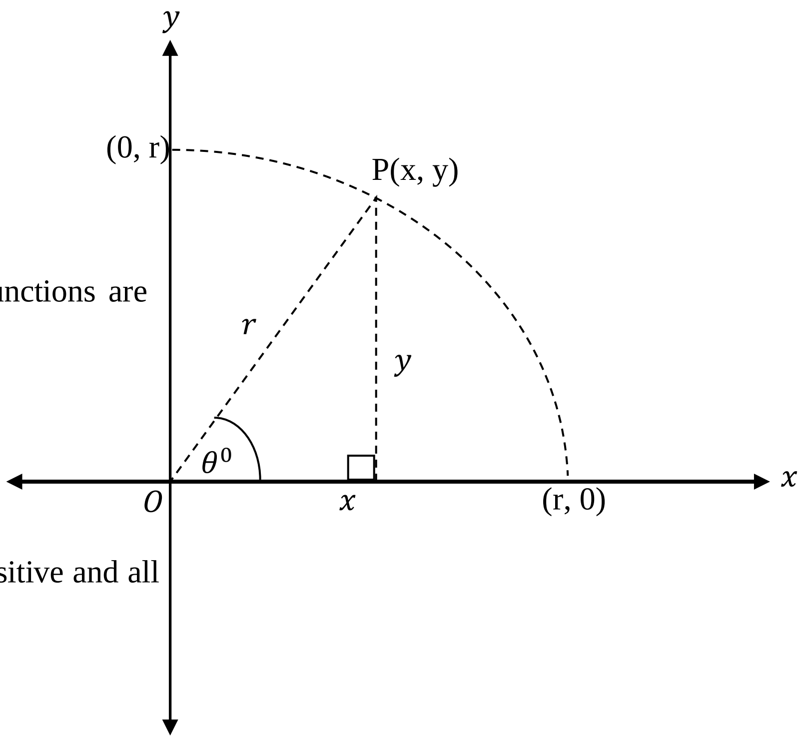


Fig. 3.5

Thus, in the first quadrant all trigonometry functions are positive.

In Second quadrant : $x < 0, y > 0 \Rightarrow sin\theta = \frac{y}{r} > 0,\ cos\theta = \frac{x}{r} < 0,\ tan\theta = \frac{y}{x} < 0.$

Thus, in the second quadrant sin function is positive and all others are negative.

In Third quadrant : $x < 0, y < 0 \Rightarrow sin\theta = \frac{y}{r} < 0, cos\theta = \frac{x}{r} < 0, tan\theta = \frac{y}{x} > 0.$

Thus, in the third quadrant all trigonometric functions are negative except tangent

In Fourth quadrant

$$: x > 0, y < 0 \Rightarrow sin\theta = \frac{y}{r} < 0, \quad cos\theta = \frac{x}{r} > 0, tan\theta = \frac{y}{x} < 0.$$

Thus, in the fourth quadrant all trigonometric functions are negative except cos.

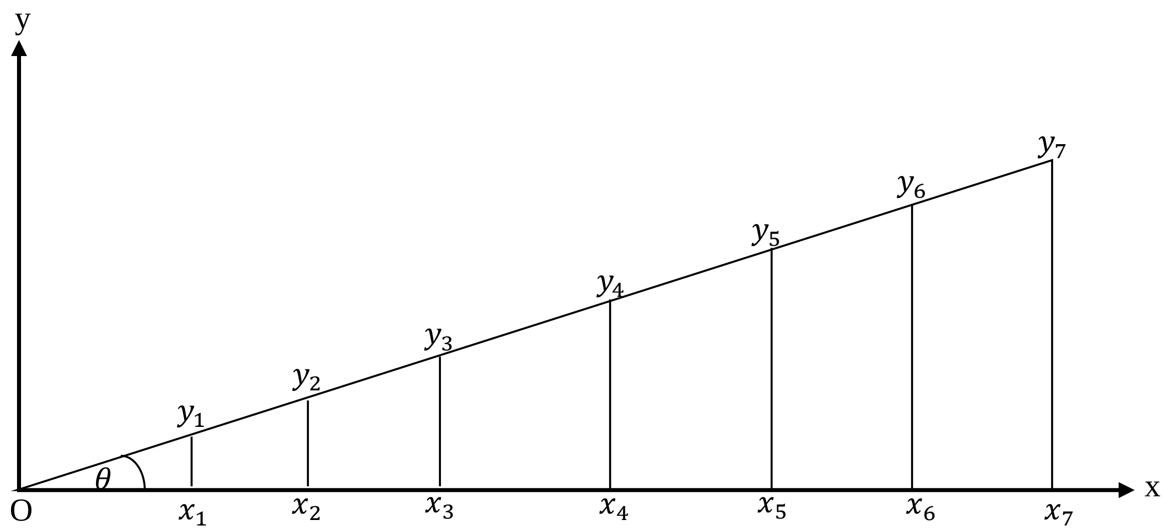


Fig. 3.6

Consider the figure above, with the same acute angle θ, measure the side of each right angled-triangle. By computing the values of the ratios $\frac{x_1y_1}{Ox_1} = \frac{x_2y_2}{Ox_2} = \frac{x_3y_3}{Ox_3} = \frac{x_4y_4}{Ox_4} = \frac{x_5y_5}{Ox_5} = \frac{x_6y_6}{Ox_6} = \frac{x_7y_7}{Ox_7}$ they are all equal to a constant.

The constant depends on the value of the angle θ. In conclusion, $\frac{length\ of\ opposite\ side}{length\ of\ adjacent\ side} = a\ constant$. The constant for the direct linear variation is called the tangent of θ and is denoted by $\tan\theta$.

$\tan\theta = \frac{length\ of\ side\ opposite\ to\ \theta}{length\ of\ side\ adjacent\ to\ \theta}$.

Same way, $\frac{x_1y_1}{Oy_1} = \frac{x_2y_2}{Oy_2} = \frac{x_3y_3}{Oy_3} = \frac{x_4y_4}{Oy_4} = \frac{x_5y_5}{Oy_5} = \frac{x_6y_6}{Oy_6} = \frac{x_7y_7}{Oy_7}$ they are also equal to a constant. The constant depends on the value of the acute angle θ. In conclusion, $\frac{length\ of\ opposite\ side}{length\ of\ the\ hypotenuse} = a\ constant$. The constant for the direct linear variation is called the sine of θ and is denoted by $\sin\theta$.

$\sin\theta = \frac{length\ of\ side\ opposite\ to\ \theta}{length\ of\ hypotenuse}$.

Similarly, $\cos\theta = \frac{length\ of\ side\ adjecent\ to\ \theta}{length\ of\ hypotenuse}$.

It will help if you have a way of recalling these definitions. One of these ways is by remembering a nonsense word: SOH TOA CAH sine is opposite over hypotenuse; tangent is opposite over adjacent and cosine is adjacent over hypotenuse. Some people remember it as SOH CAH TOA Simply changing the syllables around. Others remember it by a little verse:

Tom's Old Aunt (TOA)

Sat On Him (SOH)

Cursing At Him. (CAH). Whichever way you learn it, it will be helpful in order to remember these ratios.

$\sin\theta = \frac{y}{\sqrt{x^2-y^2}}$

$\cos\theta = \frac{x}{\sqrt{x^2-y^2}}$

$\tan\theta = \frac{y}{x}$ …………(1)

$\frac{\sin\theta}{\cos\theta} = \frac{y}{\sqrt{x^2-y^2}} \times \frac{\sqrt{x^2-y^2}}{x} = \frac{y}{x}$ ……….(2)

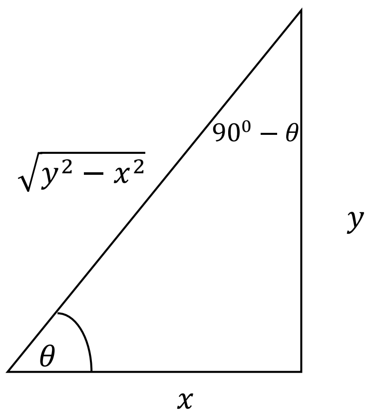


Fig. 3.7

Comparing equations (1) and (2), we can conclude that $tan\theta = \frac{sin\theta}{cos\theta}$

$\sin\theta = cos(90^0 - \theta)$
$cos\,\theta = \sin(90^0 - \theta)$

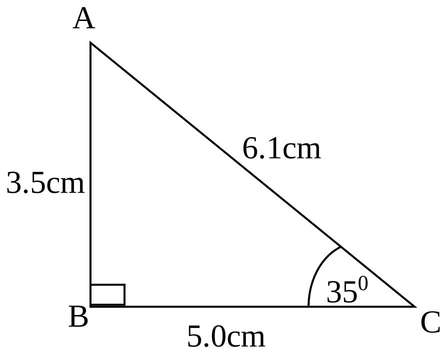


Fig. 3.8

Example 1
Find the tangent cosine of 35^0. Hence find the $\sin 55^0$

Solution
Using a sharp pencil, a ruler and a protractor, we draw a right angled-triangle ABC in which the acute angle at C is 35^0. Fig. 6.3
$|BC| = 50cm,\ |AC| = 6.1cm,\ |AB| = 3.5cm$
$\tan 35^0 = \frac{|AB|}{|BC|} = \frac{3.5}{5.0} = 0.7\ \ \tan 35^0 = 0.7$
$\cos 35^0 = \frac{|BC|}{|AC|} = \frac{5.0}{6.1} = 0.82$
$\cos\theta = sin(90^0 - \theta) \qquad \Rightarrow \cos 35^0 = \sin 55^0 = 0.82$

In Fig. 6.6, *Ox* and *Oy* are two perpendicular axes. The coordinates of P with respect to these axes are (*x*, *y*) and angle POQ = θ

By Pythagoras' theorem,
$|OP| = r = \sqrt{x^2 + y^2} \Rightarrow r^2 = x^2 + y^2$
$sin\theta = \frac{y}{r}, \quad cos\theta = \frac{x}{r}$
$cos^2\theta + sin^2 = \left(\frac{x}{r}\right)^2 + \left(\frac{y}{r}\right)^2$
$= \frac{x^2+y^2}{r^2}$
But $r^2 = x^2 + y^2$

$\Rightarrow cos^2\theta + sin^2 = \frac{x^2+y^2}{x^2+y^2} = 1$

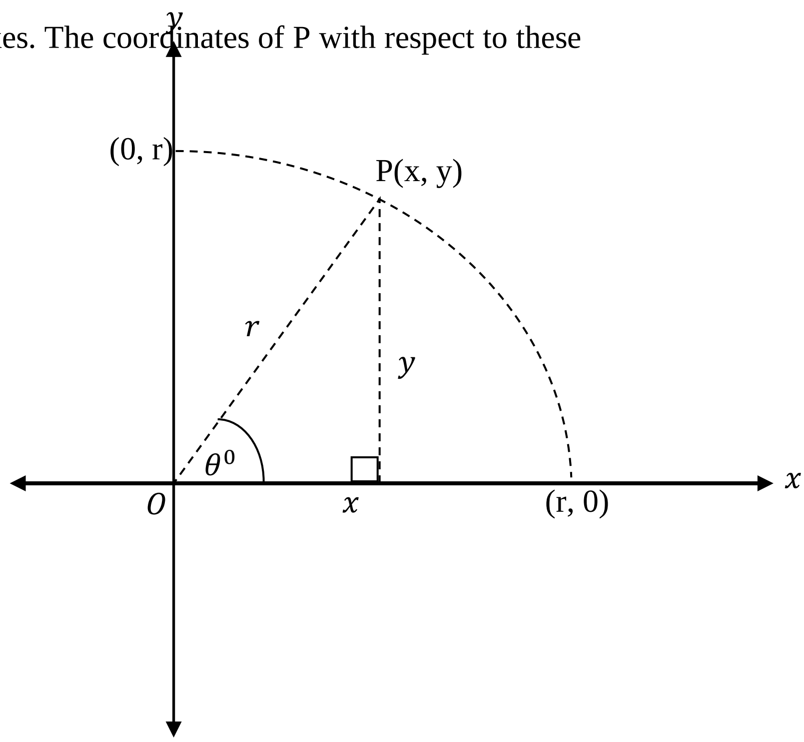


Fig. 3.9

When OP is rotated about O in the plane Oxy, from Ox to Oy and r is kept constant. We note that P lies on OX, (x, y) = (r, 0) and $\theta = 0$. Therefore

$sin0^0 = \frac{0}{r} = 0$, $cos0^0 = \frac{r}{r} = 1$ and $tan0^0 = \frac{0}{r} = 0$

When P lies on Oy, $\theta = 90^0$ and (x, y) = (0, r). Therefore

$sin90^0 = \frac{r}{r} = 1$, $cos90^0 = \frac{0}{r} = 0$ and $tan90^0 = \frac{r}{0} = undefined$

Common angles (45^0, 60^0, 30^0)

Consider the following equilateral triangle of side 2 units with an altitude constructed.

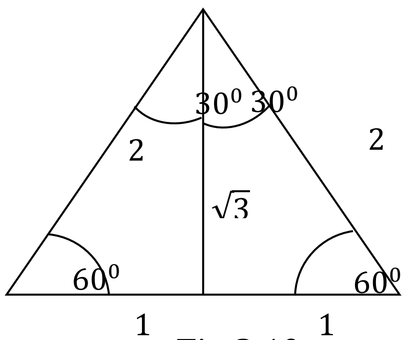
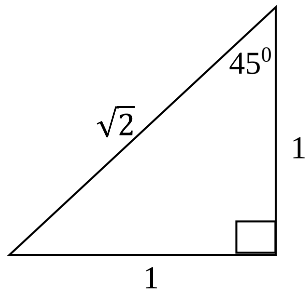


Fig.3.10

$\sin 60^0 = \frac{\sqrt{3}}{2}$, $\cos 60^0 = \frac{1}{2}$, and $tan60^0 = \sqrt{3}$

$\sin 30^0 = \frac{1}{2}$, $\cos 30^0 = \frac{\sqrt{3}}{2}$, $and\ tan30^0 = \frac{1}{\sqrt{3}}$



$\sin 45^0 = \frac{1}{\sqrt{2}}$, $\cos 45^0 = \frac{1}{\sqrt{2}}$,

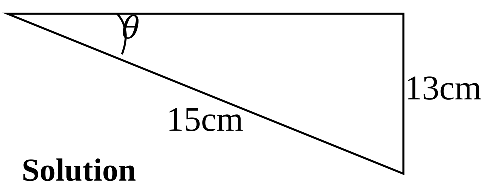
and $tan45^0 = 1$

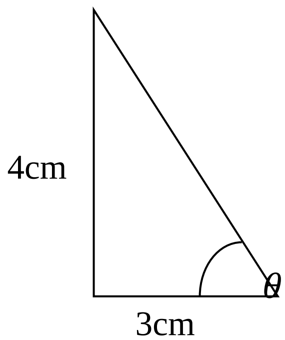
The table below gives a summary of the ratios of the special angles. You must always remember them off-head

	0°	30°	45°	60°	90°
sine	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1
cosine	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0
tangent	0	$\frac{1}{\sqrt{2}}$	1	$\sqrt{3}$	undefined

Example 3

Find θ in the following





Solution

$\sin\theta = \frac{13}{15} = 0.8667$

$\tan\theta = \frac{4}{3} = 1.3333$

$\theta = \tan^{-1}(1.3333) = 53.1^0$

$\theta = \sin^{-1}(0.8667) = 60.1^0$

Try these

1. Fig 5.9a shows a cross section of a factory roof. One side of the roof slopes at an angle of 30^0 and the other side slopes at an angle of 55^0. The total width of the roof is 5m. Calculate the height h of the roof correct to two decimal places.

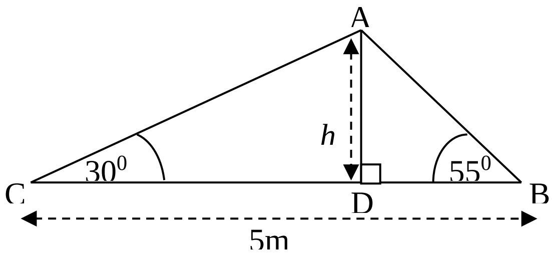


Fig. 3.10a

2. Fig. 6.8b is a rectangle ABCD and OA is perpendicular to OB, $|BC| = 2cm,\ |CD| = 6cm$ and $tanx^0 = \frac{3}{4}$. Without using calculator, find the values of $sinx^0\ and\ cosx^0$. Find also the length of OL.

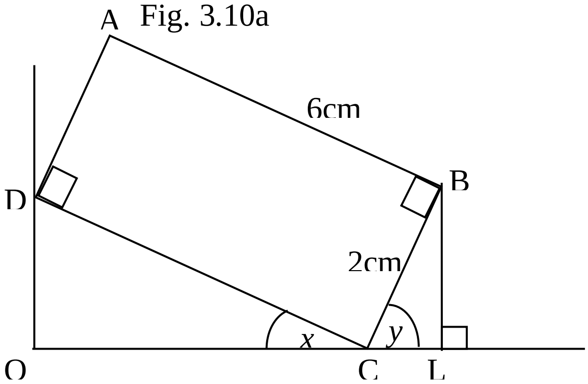


Fig. 3.10b

To Find the Trigonometric ratio of an Angle

- Sketch the angle in relation to the x and y axis
- Find the associated acute angle, ie the acute angle the radius makes with the x- axis. Find the required ratio of the angle
- Find the appropriate sign using the CAST diagram. This shows which ratios are positive in each quadrant.

Example 4

Find the value of sin 120

Solution

Sketch the 120^o, remember that in Trigonometry, angles are measured in anti-clockwise direction

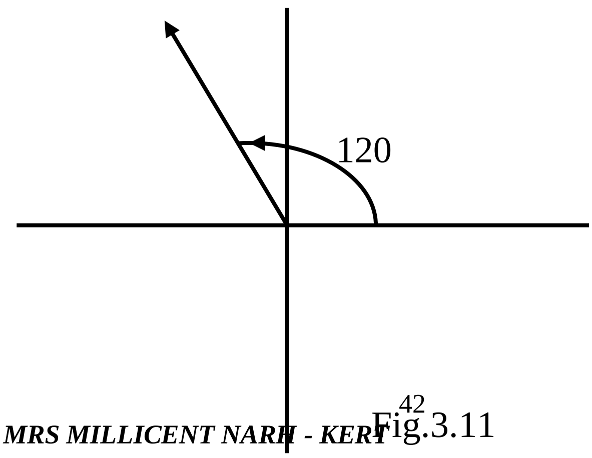


Fig.3.11

120 is in the second quadrant, therefore its sine ratio is positive. The associated acute angle with the x- axis is 60^o

Therefore, sin 120 = + sin 60 = $\frac{\sqrt{3}}{2}$

Example 5

Find the value of cos 240

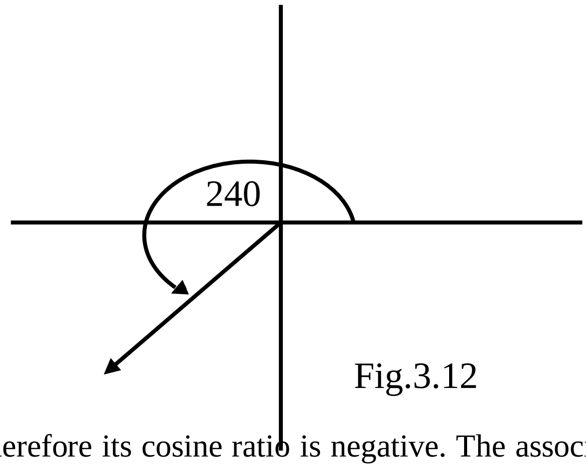


Fig.3.12

240 is in the third quadrant, therefore its cosine ratio is negative. The associated acute angle with the x- axis is 60^o

Therefore, cos 240 = -cos 60 = $-\frac{1}{2}$

Basic Trigonometric Identities

An identity is a relationship which is true for all values of the variable. The circle with centre 'O' and radius 'r' drwn on the x-y axis helps to identify some of the identities.

Draw a circle with radius 'r' and centre O as the origin, embedded on the X-Y axes as shown below;

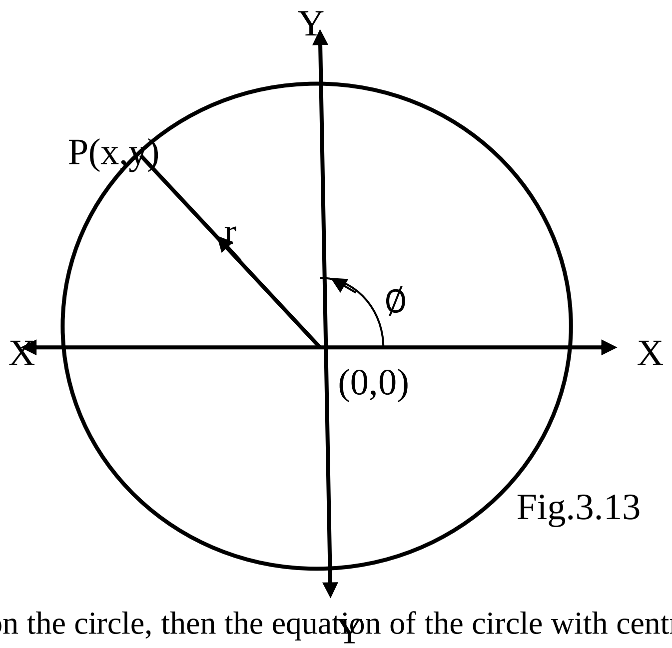


Fig.3.13

If P is the point (x,y) on the circle, then the equation of the circle with centre (0,0) and radius r is given as

$x^2 + y^2 = r^2$…………………………………..(A)

But we know that cos $\emptyset = \frac{x}{r}$ and sin $\emptyset = \frac{y}{r}$

Dividing both sides of the of the circle by r^2, we obtain

$\frac{x^2}{r^2} + \frac{y^2}{r^2} = \frac{r^2}{r^2} \Longrightarrow \left(\frac{x}{r}\right)^2 + \left(\frac{y}{r}\right)^2 = 1$

Therefore, $\boldsymbol{cos^2}\emptyset + \boldsymbol{sin^2}\emptyset = 1$………………..(1)

This is the first identity. From the first identity, we can deduce that;

$$cos^2\emptyset = 1 - sin^2\emptyset$$
$$sin^2\emptyset = 1 - cos^2\emptyset$$

Again from equation (A), $x^2 + y^2 = r^2$, dividing both sides x^2 , we get

$1 + \left(\frac{y}{x}\right)^2 = \left(\frac{r}{x}\right)^2$

$\Longrightarrow \quad 1 + (\tan\emptyset)^2 = (\sec\emptyset)^2$

Therefore, $\mathbf{1} + (\mathbf{tan}\emptyset)^2 = (\mathbf{sec}\emptyset)^{\mathbf{2}}$ ………….(2)

Similarly, if we divide the (A) by y^2, we get

$\left(\frac{x}{y}\right)^2 + 1 = \left(\frac{r}{y}\right)^2 \Longrightarrow (cos\emptyset)^2 + 1 = (cosec\emptyset)^2$

Therefore $\mathbf{1} + \boldsymbol{cot^2}\emptyset = \boldsymbol{cosec^2}\emptyset$ …………….(3)

Also, $\frac{\sin\emptyset}{\cos\emptyset} = \frac{y/r}{x/r} = \frac{y}{x} = \tan\emptyset$

Therefore, $\boldsymbol{tan}\emptyset = \frac{\mathbf{sin}\,\emptyset}{\boldsymbol{cos}\emptyset}$……………………….(4)

Converting one trigonometric ratio to another

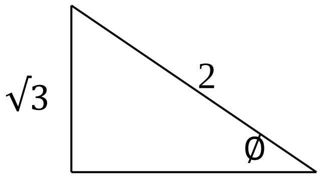
These types of problems are often solved by drawing the appropriate right-angled triangle and using Pythagoras theorem.

Example 6

If sinØ lies in the second quadrant, find cos Ø and sinØ

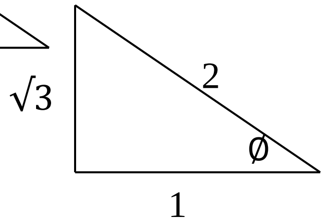
solution

We have sin Ø = $\frac{\sqrt{3}}{2}$, from the sketch,



let the unknown side be x.

By Pythagoras theorem, $2^2 = (\sqrt{3})^2 + x^2$



x = 4 - 3, x = 1

cos Ø $= \pm\frac{1}{2}$ and tan Ø = - $\sqrt{3}$ since Ø lies in the second quadrant

Example 7

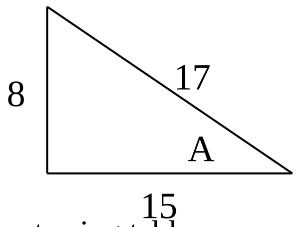
Given that sin A = $\frac{8}{17}$ and that A is an obtuse angle, find cos A and tan A without using tables

solution

Since A is in the second quadrant, it means both cosine and tangent are both negative. From the right-angled right angle, we can find the third side

$\sqrt{17^2 - 8^2} = \sqrt{225} = 15$

cosA = $-\frac{15}{17}$ and tan A = $-\frac{8}{15}$



Example 8

If 270 < Ø < 360 and cos Ø 0.6, find sin Ø and tan Ø without using tables

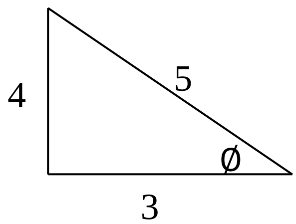
Solution

Note that 0.6 = $\frac{6}{10} = \frac{2}{5}$

Ø is in the fourth quadrant since 270 < Ø < 360 . therefore, both sine and tan are negative

$\sqrt{5^2 - 3^2} = \sqrt{16} = 4$

sinØ = $-\frac{4}{5}$ and tan Ø = $-\frac{4}{3}$



Example 9

Eliminate Ø from the equations

x = 3cos Ø and y = 2 sin Ø

solution

x = 3cos implies $\frac{x}{3}$ = cos ∅ ………………….(1)

y = 2 sin ∅ implies $\frac{y}{2}$ = sin ∅…………………(2)

$(1)^2 + (2)^2 \Longrightarrow \left(\frac{x}{3}\right)^2 + \left(\frac{y}{2}\right)^2 = cos^2 + sin^2$

But $cos^2 + sin^2 = 1$,

therefore, $\frac{x^2}{9} + \frac{y^2}{4} = 1$

Graphs of Trigonometric Functions

The graphs of the three trigonometric functions are drawn to show the behavior of each of them as x varies. Below are the graphs of y = sin x, y = cos x and y = tanx.

Graph of y = sin x

Using the scale of $-\pi < x < \pi$, the table below can be generated for the values of x and y:

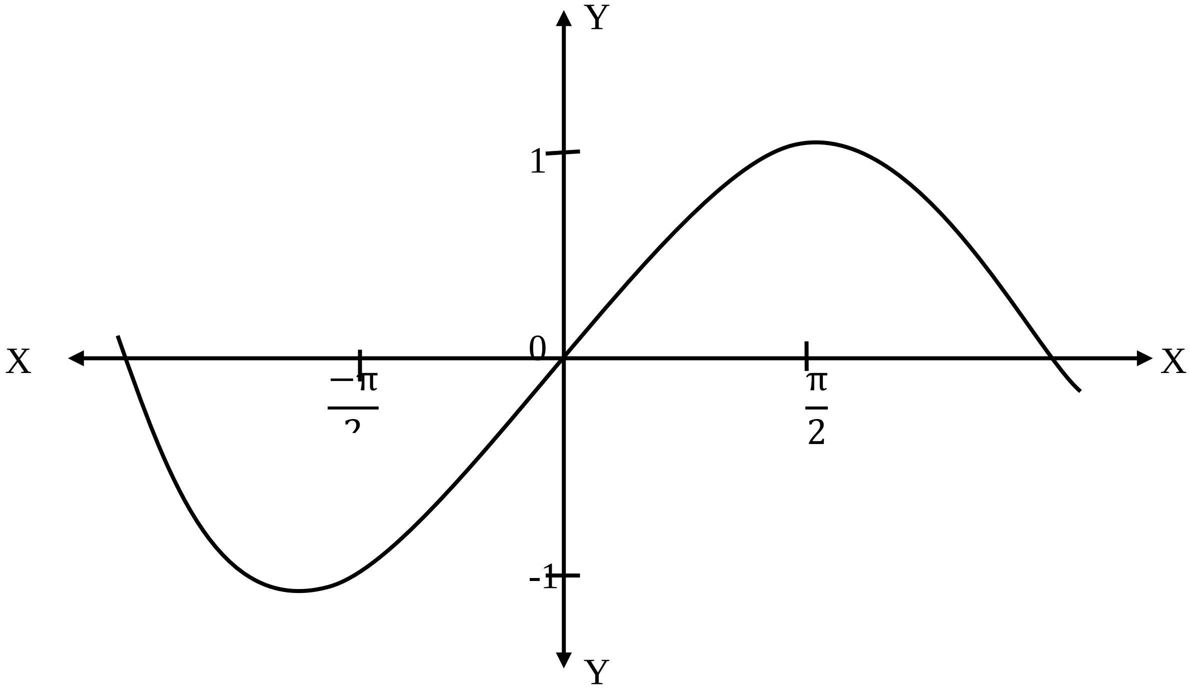
x	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π
sinx	0	0.5	71	1.0	0.87	0.71	0.5	0

Using the results sin(-x) = -sinx, we can write down the tables for the values of x between $-\pi$ and 0 as:

x	$-\pi$	$\frac{-5\pi}{6}$	$\frac{-3\pi}{4}$	$\frac{-2\pi}{3}$	$\frac{-\pi}{2}$
sinx	0	- 0.5	-0.71	-	-1.0

x	$-\pi/3$	$\frac{-\pi}{4}$	$\frac{-\pi}{6}$	0
sinx	0	- 0.5	-0.71	0

The graph corresponding to these points is as given:



Sin ∅ has values between -1 and 1 inclusive and increases from 0 to 1

Graph of y = cos x

Using the scale of $-\pi < x < \pi$, the table below can be generated for the values of x and y:

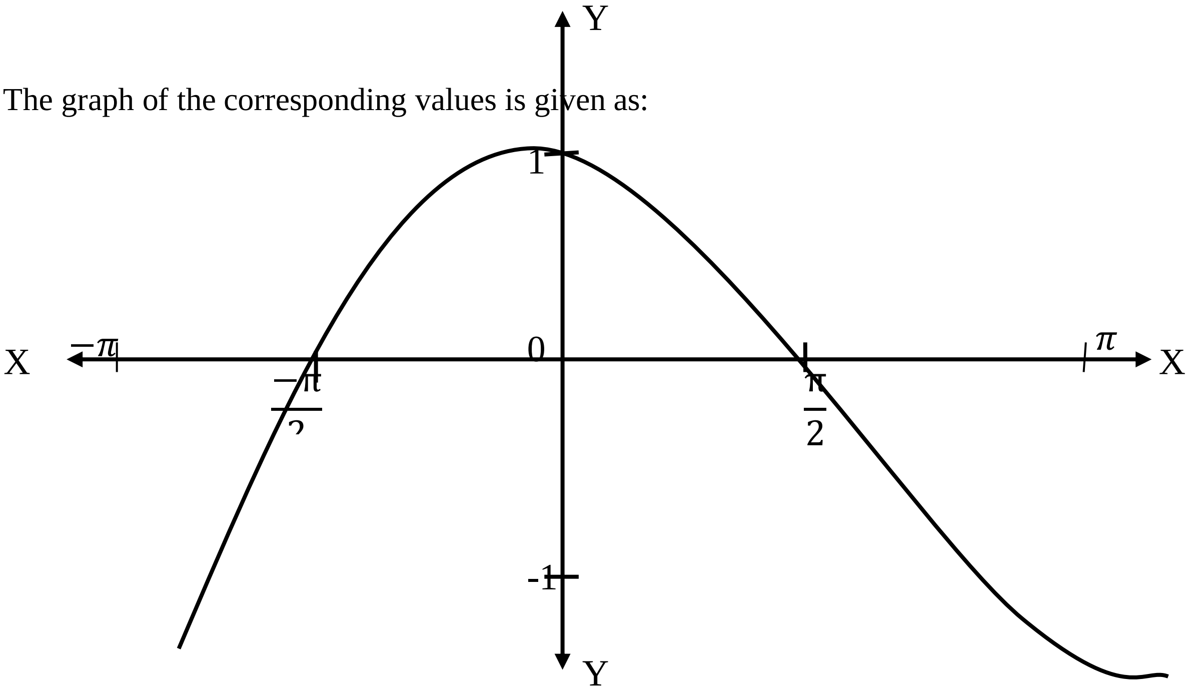
x	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π
cosx	1.0	0.87	0.71	0.5	0	-0.5	-0.71	0.87	-1.0

Using the results cos(-x) = -cosx, we can write down the tables for the values of x between $-\pi$ and 0 as:

x	$-\pi$	$\frac{-5\pi}{6}$	$\frac{-3\pi}{4}$	$\frac{-2\pi}{3}$	$\frac{-\pi}{2}$
cosx	- 1.0	-	-0.71	-0.5	0

x	$-\pi/3$	$\frac{-\pi}{4}$	$\frac{-\pi}{6}$	0
sinx	- 0.5	0.71	0.87	1

The graph of the corresponding values is given as:



Cos ∅ has values between -1 and 1 inclusive and decreases from1 to 0

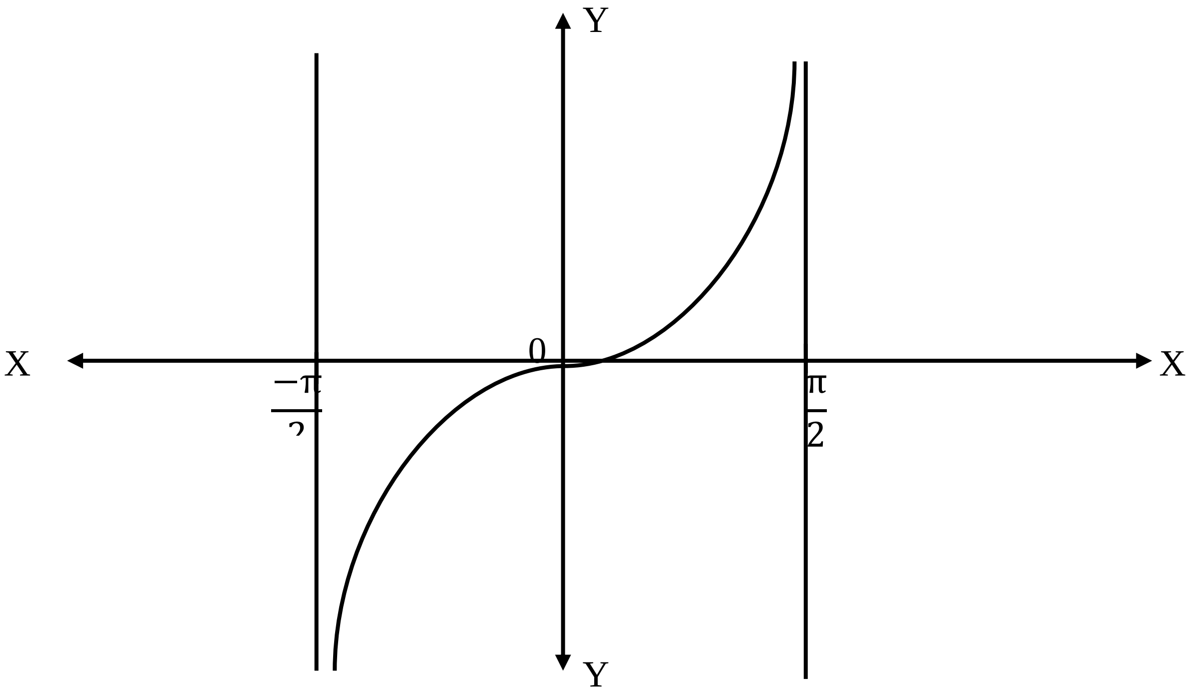
Graph of y = tan x

Using the scale of $-\pi/2$ < x < $\pi/2$, the table below can be generated for the values of x and y:

Note that tanx does not exist for x = $\pm\pi/2$. Since sinx increases from 0 to 1 and cosx decreases from1 to 0, and tan x = $\frac{\text{sinx}}{\text{cosx}}$, tanx will increase indefinitely as x (staring from the value of 0) approaches $\pi/2$. Similarly, as x (starting from the value 0) approaches $-\pi/2$, tanx decreases indefinitely.

x	$-\frac{\pi}{3}$	$-\frac{\pi}{4}$	$-\frac{\pi}{6}$	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$
tanx	-1.73	-1.0	-0.58	0	0.58	1.0	1.73

The graph of the corresponding values is given as:



Remember that tan 90^o and tan 270^o do no exist.

General Solutions of trigonometric equations

- The general Solutions of **sin $\boldsymbol{\theta}$** = a is

$\theta = n\pi + (-1)^n\alpha$, where $\alpha = sin^{-1}\alpha$ in radians. Or $\theta = 180n + (-1)^n\alpha$, where $\alpha = sin^{-1}\alpha$ in degrees.

- The general Solutions of **cos $\boldsymbol{\theta}$** = b is

$\theta = 2n\pi \pm \beta$, where $\beta = cos^{-1}b$ in radians. Or $\theta = 360n \pm \beta$ where $\beta = cos^{-1}b$ in degrees.

- The general Solutions of **tan $\boldsymbol{\theta}$** = c is

$\theta = n\pi + \gamma$, where $\gamma = tan^{-1}c$ in radians. Or $\theta = 180n + \gamma$ where $\gamma = tan^{-1}c$ in degrees.

Note that n = 0,1,2,3,……

Example 10

Find the general solution of $\sin\emptyset = 1$

Solution

The principal solution of the equation $\sin\emptyset = 1$ is $\emptyset = \frac{\pi}{2}$ or 90^o

So the general solution is;

$\theta = n\pi + (-1)^n\frac{\pi}{2}$ in radians or $\theta = 180n + (-1)^n 90^o$ in degrees.

Example 11

Find the general solution of $\cos\emptyset = \frac{1}{2}$

Solution

The principal solution of the equation $\cos\emptyset = \frac{1}{2}$ is $\theta = \frac{\pi}{3}$ or 60^o

So the general solution is; $\theta = 2n\pi \pm \frac{\pi}{3}$ in radians or $\theta = 360n \pm 60^o$ in degrees.

Example 12

Find the general solution of $tanx = 1$

Solution

The principal solution of the equation $\tan x = 1$ is $x = \frac{\pi}{4}$ or 45^o

So the general solution is; $x = n\pi \pm \frac{\pi}{4}$ in radians or $x = 180n \pm 45^o$ in degrees.

Example 13

Find the general solution of $2cos\frac{1}{2}x = 1$

Solution

$2cos\frac{1}{2}x = 1$ implies $cos\frac{1}{2}x = \frac{1}{2}$, let $\frac{1}{2}x = \theta$

The principal solution of the equation $\cos\emptyset = \frac{1}{2}$ is $\emptyset = \frac{\pi}{3}$ or 60^o

So the general solution is; $\emptyset = 2n\pi \pm \frac{\pi}{3}$ $\frac{1}{2}x \Longrightarrow 2n\pi \pm \frac{\pi}{3}$

Hence the general solution is $x = 4n\pi \pm \frac{2\pi}{3}$ in radians or $\frac{1}{2}x = 360n \pm 60^o$

$\Longrightarrow x = 720n \pm 120$ in degrees.

Try this

Draw the graph of y = sinx +cosx for values of x from 0^o to 360^o using intervals of 30. Use your graph to find:

- the values of x correct to the nearest degree for which sinx + cosx = 0.75
- the minimum and maximum values of y, stating the values of x for which they occur.

STRAND 4: VECTORS AND BEARING:

LEARNING, TEACHING AND APPLYING

Vectors

Quantities that are determined only by magnitude, i.e., length, mass, temperature, etc., are called scalars. A vector is a line segment (with magnitude) and an assigned direction. An arrow is used to specify the direction. Vector $\overrightarrow{AB}$ has initial point A and terminal point B. The magnitude or length of the vector is the length of the segment AB and is denoted by $\overrightarrow{|AB|}$. Two vectors are equal if they have equal magnitude and the same direction.

A scalar is a number which expresses quantity. Scalars may or may not have units associated with them. Examples: mass, volume, energy, money

A vector is a quantity which has both magnitude and direction. The magnitude of a vector is a scalar. Examples: Displacement, velocity, acceleration, electric field etc.

Vectors are denoted as a symbol with an arrow over the top: $\vec{X}$

Vectors can be written as a magnitude and direction: $(20km,\ \ 150^0)$

Vector Representation

Vectors are represented by an arrow pointing in the direction of the vector. For example, we represent the movement from O to A in a given plane as $\overrightarrow{OA}$, where O is the starting or initial point and A is the destination point. The arrow gives the direction of the vector.

The length of the vector represents the magnitude of the vector.

Note that, the length of the arrow does not necessarily represent a length.

a) The vector $\overrightarrow{OA}$ may be represented by (magnitude and direction) as stated before. See Fig. 4.1

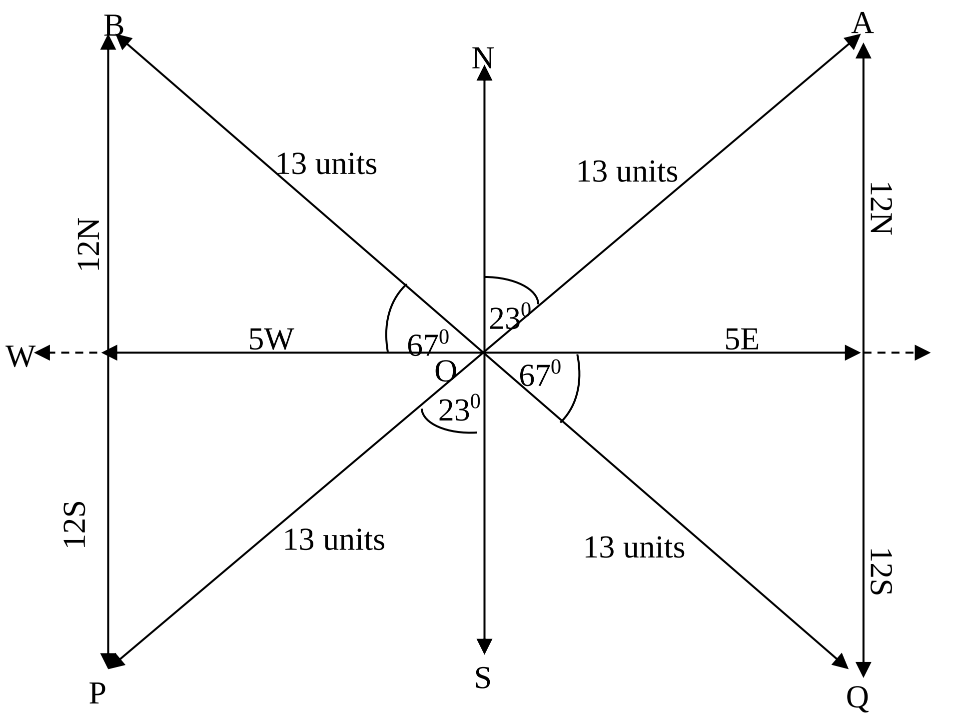


Fig. 4.1

From fig. 4,1, $\overrightarrow{OA} = (13\,units,\ 023^0)$. This form is called directional vector since it specifies the direction of movement from O to A. similarly, $\overrightarrow{OB} = (13\,units,\ 337^0)$, $\overrightarrow{OP} = (13\,units,\ 203^0)$ and $\overrightarrow{OQ} = (13\,units,\ 157^0)$

b) The vector , $\overrightarrow{OA}$ can also be represented by two rational numbers called components. i.e. , $\overrightarrow{OA} = \begin{pmatrix} 5 \\ 12 \end{pmatrix}$, where 5 is the x-component and 12 is the y-component. The x-component indicates how far east B is from A and whilst the y-component indicate how far north B is from A. In other words, the point B is 5 units to the east of A and 12 units to the north of A. Vectors in component form are also called column vectors.
Negative sign is assign to indicate that the movement is either to the west or the south. Example, $\overrightarrow{OB} = \begin{pmatrix} 5W \\ 12N \end{pmatrix} = \begin{pmatrix} -5 \\ -12 \end{pmatrix}$, and $\overrightarrow{OQ} = \begin{pmatrix} 5E \\ 12S \end{pmatrix} = \begin{pmatrix} -5 \\ -12 \end{pmatrix}$

Magnitude bearing form $\underline{(\boldsymbol{r},\ \boldsymbol{\theta^0})}$

A quantity expressed in the for (magnitude, direction) is called a directional vector. The position of the point B from A is denoted by $\overrightarrow{AB}$. Note that

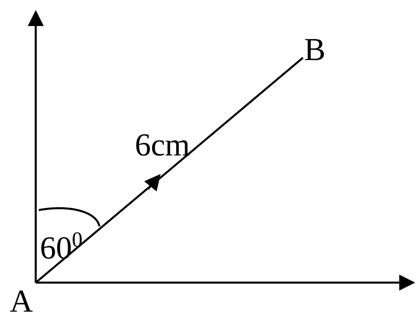


Fig. 4.2

we write the starting point of the vector before the destination point.

If the magnitude or length $|AB|$ of the vector $\overrightarrow{AB}$ is 6cm and the bearing of the point B from A is 060^0, as in Fig 4.2, then $\overrightarrow{AB} = (6cm,\ 060^0)$. In general, if $|AB| = r$ and the bearing of B from A is θ, then the vector $\overrightarrow{AB}$ may be expressed in the (magnitude, direction form as $\overrightarrow{AB} = (r, \theta)$.

Component form $\begin{pmatrix} x \\ y \end{pmatrix}$

The vector $\overrightarrow{AB}$ in Fig. 4.3 has magnitude of 5cm and the bearing of B from A is 50^0. The directional vector $\overrightarrow{AB}$ is therefore written as $(5cm,\ 50^0)$. When we use the Cartesian coordinates system to describe the vector $\overrightarrow{AB}$, and state the x-component and y-component between A and B, B is 3cm to the east of A and 4cm to the north of A. therefore, for the movement from A to B, we write $\overrightarrow{AB} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$, where 3cm is measured along the x-axis and 4cm along the y-axis from A to B.

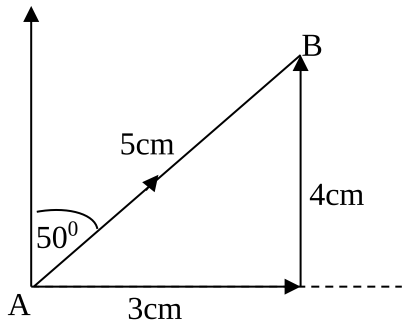


Fig. 4.3 a

If however, the position of B is 3cm to the west of A and 4 com to the north of A as shown in Fig 4.3b, then the movement from A to B is written as $\overrightarrow{AB} = \begin{pmatrix} -3 \\ 4 \end{pmatrix}$.

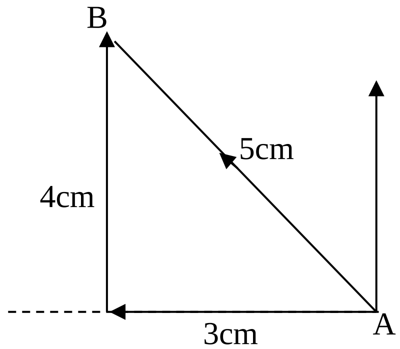


Fig. 4.3 b

In Fig. 7.18c, the position of B is 3cm to the west of A and 4cm to the south of A. Therefore, $\overrightarrow{AB} = \begin{pmatrix} -3 \\ -4 \end{pmatrix}$.

Vectors in component form are also called column vectors. The movement to the west is a negative scalar multiple of a movement to the east. Similarly, a movement to the south is a negative scalar multiple of a movement to the north.

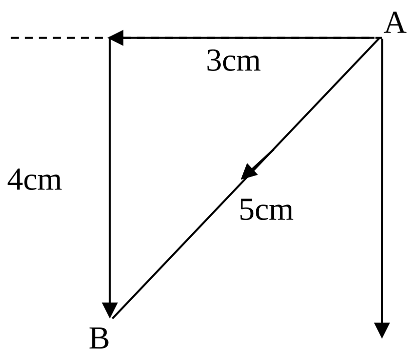


Fig. 4.3 c

Magnitude of a column vector.

The magnitude of a vector say $\overrightarrow{AB}$, is the length of the line segment $\overline{AB}$. The magnitude of a vector, say $\overrightarrow{AB}$ is denoted by $|\overrightarrow{AB}|$or $|AB|$. If $\overrightarrow{AB} = \begin{pmatrix} x \\ y \end{pmatrix}, then\ |AB| = \sqrt{x^2 + y^2}$

Example 14

Find the magnitude of the following vectors

a) $\overrightarrow{AB} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$

b) $\overrightarrow{PQ} = \begin{pmatrix} 10 \\ 24 \end{pmatrix}$

c) $\overrightarrow{XY} = \begin{pmatrix} 15 \\ 8 \end{pmatrix}$

Solution

a) $\overrightarrow{AB} = \sqrt{3^2 + 4^2} = \sqrt{9 + 16} = \sqrt{25} = 5$

b) $\overrightarrow{PQ} = \sqrt{10^2 + 24^2} = \sqrt{100 + 576} = \sqrt{676} = 26$

c) $\overrightarrow{XY} = \sqrt{15^2 + 8^2} = \sqrt{225 + 64} = \sqrt{289} = 17$

Zero vectors

A vector is said to be zero vector if its components are both zero. Example $a = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

Addition and subtraction of vectors

Resultant of vectors:

The sums of two vectors represent the resultant of the two vectors. The resultant is drawn from the tail of the first to the head of the last vector. Example the resultant of the vectors $\overrightarrow{AB}$ and $\overrightarrow{BC}$ is $\overrightarrow{AC}$. Therefore

$\overrightarrow{AB}$ and $\overrightarrow{BC}$ is $\overrightarrow{AC}$

$\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC}$

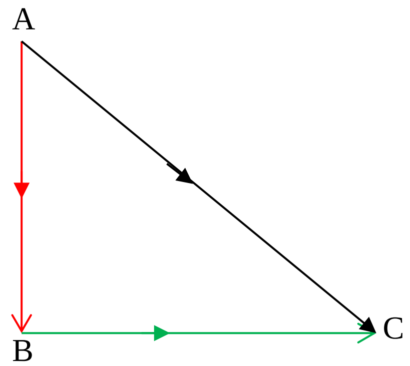


Fig. 4.4

Resultant vectors in component form:

Given $\overrightarrow{AB} = a = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$, $\overrightarrow{BC} = b = \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$ and $c = \overrightarrow{AC}$, it follows that

$c = a + b = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} x_1 + x_2 \\ y_1 + y_2 \end{pmatrix}$ (Add the corresponding components

Example 15

If $a = \begin{pmatrix} 3 \\ -4 \end{pmatrix}$ and $b = \begin{pmatrix} 5 \\ 6 \end{pmatrix}$, find $a + b$

Solution

$a + b = \begin{pmatrix} 3 \\ -4 \end{pmatrix} + \begin{pmatrix} 5 \\ 6 \end{pmatrix} = \begin{pmatrix} 3 + 5 \\ -4 + 6 \end{pmatrix} = \begin{pmatrix} 8 \\ 2 \end{pmatrix}$

Example 16

If $a = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$ and $b = \begin{pmatrix} -6 \\ 5 \end{pmatrix}$, find $a + b$

Solution

$a + b = \begin{pmatrix} 2 \\ 3 \end{pmatrix} + \begin{pmatrix} -6 \\ 5 \end{pmatrix} = \begin{pmatrix} 2 - 6 \\ 3 + 5 \end{pmatrix} = \begin{pmatrix} -4 \\ 8 \end{pmatrix}$

Parallel and Perpendicular vectors

Two non-zero vectors are said to be parallel if one is a scalar multiple of the other. If the scalar multiple is positive, it implies the vectors are in the same direction but if the scalar multiple is negative, it implies they have opposite directions. Example $\begin{pmatrix}-4\\2\end{pmatrix}$ and $\begin{pmatrix}-8\\4\end{pmatrix}$ are parallel vectors.

A vector is said to be perpendicular to another if found to have rotated 90^0 or 270^0. Example the vector $\begin{pmatrix}2\\3\end{pmatrix}$ is perpendicular to the vector $\begin{pmatrix}-3\\2\end{pmatrix}$ and $\begin{pmatrix}3\\-2\end{pmatrix}$ or their multiples $\begin{pmatrix}-3k\\2k\end{pmatrix}$ and $\begin{pmatrix}3k\\-2k\end{pmatrix}$ where k is a positive number.

The vectors $\begin{pmatrix}y\\-x\end{pmatrix}$ and $\begin{pmatrix}-y\\x\end{pmatrix}$ or their positive scalar multiples $\begin{pmatrix}ky\\-kx\end{pmatrix}$ and $\begin{pmatrix}-ky\\kx\end{pmatrix}$ are perpendicular to the vector $\begin{pmatrix}x\\y\end{pmatrix}$.

Resultant vectors in directional form.

Vectors in directional form cannot be added in the same way the component vectors are added. This is because vectors in this form are represented by both magnitude and direction and in carrying such an operation, the magnitudes and the directions need to be taken into account. The resultant in this case can be determined geometrically or by converting the directional vectors to component vectors.

Geometric Approach:

Example 17

By drawing, find the resultant $\overrightarrow{AC}$ of the vectors $\overrightarrow{AB} = (5cm,\ 055^0)$ and $\overrightarrow{BC} = (4.5cm, 160^0)$

Solution

1. Choose a point A on a plain paper
2. Draw the directed line segment $\overline{AB}$ and locate the point B as in Fig. 4.5
3. Draw the directed line segment $\overline{BC}$ and locate C
4. Measure the magnitude and bearing of $\overline{AC}$. This is the resultant of $\overrightarrow{AB}$ and $\overrightarrow{BC}$.
   The bearing of C from A = $042^0 + 053^0 = 095^0$ (to the nearest degree). The distance from A to C = 10cm.
   Therefore $\overrightarrow{AC} = (5.9cm, 103^0)$

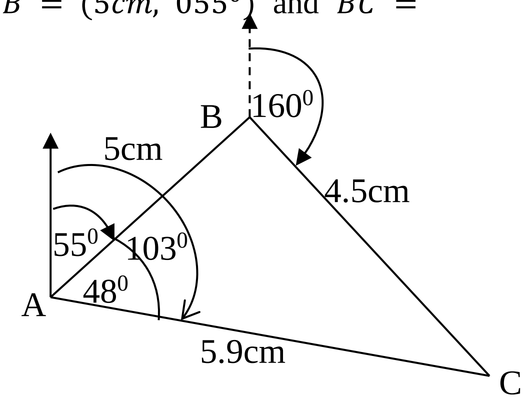


Fig.4.5

Negative Vectors

The negative of a vector are just the same vector but in the opposite direction. Example if $y = \overrightarrow{AB}$, then $-y = \overrightarrow{BA}$

Negative vectors in component form

If $\overrightarrow{AB} = \begin{pmatrix}x\\y\end{pmatrix}$, then the negation of $\overrightarrow{AB}$ is given by $\overrightarrow{BA} = -\overrightarrow{AB} = \begin{pmatrix}-x\\-y\end{pmatrix}$

Example If $\overrightarrow{AB} = \begin{pmatrix}-5\\6\end{pmatrix}$ and $\overrightarrow{BA} = \overrightarrow{-AB} = \begin{pmatrix}5\\-6\end{pmatrix}$

If $\overrightarrow{PQ} = \begin{pmatrix} 7 \\ 5 \end{pmatrix}$ then $\overrightarrow{QP} = -\overrightarrow{PQ} = \begin{pmatrix} -7 \\ -5 \end{pmatrix}$

Negative vector in directional form

If $x = (5N, 032^0)$ then $-x = (5N, 032^0 + 180^0)$ i.e. $-x = (5N, 212^0)$

If $x = (6km, 320^0)$ then $-x = (6km, 320^0 - 180^0)$ i.e. $-x = (6km, 140^0)$

If $\overrightarrow{AB} = (r, \theta)$ and $0^0 < \theta < 180^0$ then $\overrightarrow{BA} = -\overrightarrow{AB} = (r, \theta + 180^0)$

If $\overrightarrow{AB} = (r, \theta)$ and $180^0 < \theta < 360^0$ then $\overrightarrow{BA} = -\overrightarrow{AB} = (r, \theta - 180^0)$

Example 18

If $\overrightarrow{AB} = (8km, 085^0)$ then $\overrightarrow{BA} = -\overrightarrow{AB} = (8km,\ 085^0 + 180^0) = (8km,\ 265^0)$

If $\overrightarrow{AB} = (8km, 285^0)$ then $\overrightarrow{BA} = -\overrightarrow{AB} = (8km,\ 285^0 - 180^0) = (8km,\ 105^0)$

Subtraction of Vectors

The difference of two vectors a-b is equal to $a + (-b)$. The difference is therefore seen as the resultant of a and $-b$.

The resultant is drawn from the tail of the first to the head of the last vector.

Subtraction of vectors in component form.

If $a = \begin{pmatrix} w \\ x \end{pmatrix}$ and $b = \begin{pmatrix} y \\ z \end{pmatrix}$, then $a - b = \begin{pmatrix} w \\ x \end{pmatrix} - \begin{pmatrix} y \\ z \end{pmatrix} = \begin{pmatrix} w - y \\ x - z \end{pmatrix}$

Example 19

If $a = \begin{pmatrix} 8 \\ -10 \end{pmatrix}$ and $b = \begin{pmatrix} 6 \\ 12 \end{pmatrix}$, find $a - b$

Solution

$a - b = \begin{pmatrix} 8 \\ -10 \end{pmatrix} - \begin{pmatrix} 6 \\ 12 \end{pmatrix} = \begin{pmatrix} 2 \\ -22 \end{pmatrix}$

Subtraction of vectors in direction form.

If $x = (a, \mu)$ and $y = (b, \gamma)$ then $x - y = x \pm y = (a, \mu) + (b, \gamma \pm 180^0)$

Example 20

By drawing, find the resultant $\overrightarrow{AC}$ of the vectors $\overrightarrow{AB} = (40km, 320^0)$ and $\overrightarrow{CB} = (80km, 050^0)$

Solution

$\overrightarrow{AC} = \overrightarrow{AB} - \overrightarrow{CB} = \overrightarrow{AB} + \overrightarrow{BC}$

$\overrightarrow{BC} = -(80\text{km}, 050^0) = (80km, 180^0 + 050^0) = (80km, 230^0)$

$\therefore \overrightarrow{AC}(40km, 320^0) + (80km, 230^0)$

Use the steps below to draw the required figure.

1. On a plain sheet, choose point A
2. Using a scale of 2cm to 10km, draw directed line segment $\overline{AB}$ and locate the point B as shown in Fig. 8.6

3. Draw the directed line segment $\overline{BC}$ and locate the point C.
4. Measure the magnitude and bearing of $\overrightarrow{AC}$. This is the resultant of $\overrightarrow{AB}$ and $\overline{BC}$
   The distance of C from A = 89.4km
   The bearing of C from A = 320^0 – 0630 = 2570
   $\therefore \overrightarrow{AC} = (89.4km, 257^0)$ we can as well write $\overrightarrow{AB} - \overrightarrow{CB} = \overrightarrow{AC}$
   $(40km, 320^0) - (80km, 230^0) = (89.4km, 257^0)$

We could also resolve the vectors into its components and use them to determine the vector $\overrightarrow{AC}$

Example 21

1) If $\overrightarrow{AB} = \begin{pmatrix} 5 \\ 6 \end{pmatrix}$ and $\overrightarrow{BC} = \begin{pmatrix} -6 \\ 7 \end{pmatrix}$, find
   i. $\overrightarrow{AC}$
   ii. $|AC|$
2) If $\overrightarrow{AB} = \begin{pmatrix} 3 \\ 5 \end{pmatrix}$ and $\overrightarrow{CB} = \begin{pmatrix} 6 \\ 2 \end{pmatrix}$, evaluate $\overrightarrow{CA}$
3) Find $\overrightarrow{PS}$ given that $\overrightarrow{QP} = \begin{pmatrix} -2 \\ -1 \end{pmatrix}$, $\overrightarrow{QR} = \begin{pmatrix} -3 \\ 4 \end{pmatrix}$ and $\overrightarrow{SR} = \begin{pmatrix} -5 \\ 3 \end{pmatrix}$.

Solution

1)
   i. $\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC} = \begin{pmatrix} 5 \\ 6 \end{pmatrix} + \begin{pmatrix} -6 \\ 7 \end{pmatrix} = \begin{pmatrix} -1 \\ 13 \end{pmatrix}$
   ii. $|AC| = \sqrt{-1^2 + 13^2} = \sqrt{170} = 13.0384 \approx 13$
2) $\overrightarrow{CA} = \overrightarrow{CB} + \overrightarrow{BA}$
   $\overrightarrow{AB} = \begin{pmatrix} 3 \\ 5 \end{pmatrix} \Rightarrow \overrightarrow{BA} = -\overrightarrow{AB} = \begin{pmatrix} -3 \\ -5 \end{pmatrix}$
   $\overrightarrow{CA} = \begin{pmatrix} 6 \\ 2 \end{pmatrix} + \begin{pmatrix} -3 \\ -5 \end{pmatrix} = \begin{pmatrix} 3 \\ -3 \end{pmatrix}$

3) $\overrightarrow{PQ} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \overrightarrow{QR} = \begin{pmatrix} -3 \\ 4 \end{pmatrix}$ *and* $\overrightarrow{RS} = \begin{pmatrix} 5 \\ -3 \end{pmatrix}$.
4) $\overrightarrow{PS} = \overrightarrow{PQ} + \overrightarrow{QR} + \overrightarrow{RS} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} + \begin{pmatrix} -3 \\ 4 \end{pmatrix} + \begin{pmatrix} 5 \\ -3 \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \end{pmatrix}$

Scalar multiple of a vector

Vectors in component form:

If a vector $\overrightarrow{AB} = \begin{pmatrix} x \\ y \end{pmatrix}$ and k is a scalar, then the scalar multiple of $\overrightarrow{AB}$ by k is given by $k\,\overrightarrow{AB} = k\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} kx \\ ky \end{pmatrix}$

Example 21

1) If $p = \begin{pmatrix} 4 \\ 5 \end{pmatrix}$ and $q = \begin{pmatrix} -4 \\ 8 \end{pmatrix}$, find $4p - 3q$
2) What can you say about your answer in (1) and the vector $\begin{pmatrix} -14 \\ 2 \end{pmatrix}$

Solution

1) If $p = \begin{pmatrix}4\\5\end{pmatrix} \Rightarrow 4\text{p} = \begin{pmatrix}16\\20\end{pmatrix}$ and $\text{q} = \begin{pmatrix}-4\\8\end{pmatrix} \Rightarrow 3\text{q} = \begin{pmatrix}-12\\24\end{pmatrix}$,

   $\therefore 4p - 3q = \begin{pmatrix}16\\20\end{pmatrix} - \begin{pmatrix}-12\\24\end{pmatrix} = \begin{pmatrix}28\\-4\end{pmatrix}$

2) $4p - 3q = \begin{pmatrix}28\\-4\end{pmatrix} = -2\begin{pmatrix}-14\\2\end{pmatrix}$

   Hence the vector $\begin{pmatrix}28\\-4\end{pmatrix}$ is parallel but opposite in direction to $\begin{pmatrix}-14\\2\end{pmatrix}$

Vector in directional form

If the vector $\overrightarrow{AB} = (\text{r}, \theta)$ and k is a positive scalar, then $k\,\overrightarrow{AB} = (\text{kr}, \theta)$

Example 22

Given $\overrightarrow{\text{AB}} = (5\text{cm}, 030^0)$ it implies that $3\overrightarrow{\text{AB}} = (3 \times 5\text{cm},\ 030^0) = (15\text{cm},\ 030^0)$

The magnitude is 3 times that of $\overrightarrow{\text{AB}}$ but the direction remains the same.

When a vector$\overrightarrow{AB} = (\text{r}, \theta)$ is multiplied by a negative scalar, the direction is opposite to that of $\overrightarrow{AB}$. That is if $k > 0$, then -$k\,\overrightarrow{AB} = (\text{kr}, \pm 180^0)$

Example 23

1) Given $\overrightarrow{\text{AB}} = (5\text{cm}, 030^0)$ it implies that $-3\overrightarrow{\text{AB}} = (3 \times 5\text{cm},\ 030^0 + 180^0) = (15\text{cm},\ 210^0)$
2) Given $\overrightarrow{\text{AB}} = (5\text{cm}, 330^0)$ it implies that $-3\overrightarrow{\text{AB}} = (3 \times 5\text{cm},\ 330^0 - 180^0) = (15\text{cm},\ 150^0)$

Example 24

If $\underline{a} = \begin{pmatrix}3\\2\end{pmatrix}$, $\underline{b} = \begin{pmatrix}u\\-4\end{pmatrix}$ *and* $\underline{c} = \begin{pmatrix}-2\\v\end{pmatrix}$, find the values of u and v such that $2\underline{a} = \underline{b} + \underline{c}$

Solution

$2\underline{a} = \underline{b} + \underline{c} \Leftrightarrow 2\begin{pmatrix}3\\2\end{pmatrix} = \begin{pmatrix}u\\-4\end{pmatrix} + \begin{pmatrix}-2\\v\end{pmatrix} \Rightarrow \begin{pmatrix}6\\4\end{pmatrix} = \begin{pmatrix}u-2\\-4+v\end{pmatrix}$

$6 = u - 2 \Rightarrow u = 6 + 2 = 8$

$4 = -4 + v \Rightarrow v = 4 + 4 = 8.$

$u = 8$ and $v = 8$

Position Vector

In general, a vector has no specific location in space. However, if $a = \overrightarrow{OA}$, where O is a fixed origin, then a is referred to as the position vector of A relative to O.

Expressing two given points as a vector.

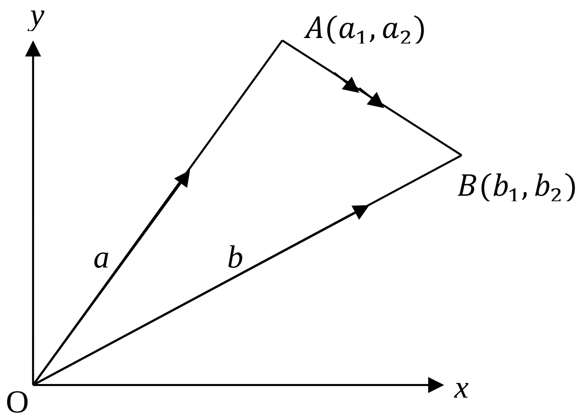


Fig.4.6

Fig. 8.6 shows position vectors ***a*** and ***b*** of the points A and B respectively. The vector

$\overrightarrow{AB} = \overrightarrow{OA} + \overrightarrow{OB} = -a + b = b - a$

$= \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} - \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} b_1 - a_1 \\ b_2 - a_2 \end{pmatrix}$

$\overrightarrow{AB}$ = position verctor B – position vector A

Example 25

1) Find the vector $\overrightarrow{AB}$ if $a = \begin{pmatrix} 4 \\ -3 \end{pmatrix}$ and $b = \begin{pmatrix} 7 \\ 10 \end{pmatrix}$

Solution

1) $\overrightarrow{AB} = b - a = \begin{pmatrix} 7 \\ 10 \end{pmatrix} - \begin{pmatrix} 4 \\ -3 \end{pmatrix} = \begin{pmatrix} 3 \\ 13 \end{pmatrix}$

The position vector of the mid-point of a line segment

Consider the line joining the points A and B with position vectors ***a*** and ***b*** respectively relative to the origin O. Let M be the midpoint of AB with position vector ***m*** relative to the same origin O. From the Fig 7.7:

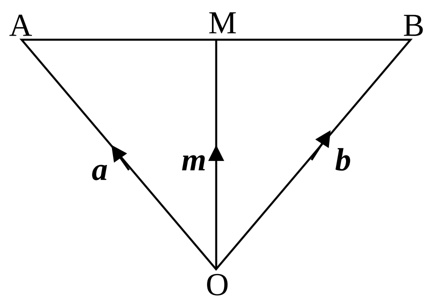


Fig. 4.7

$\overrightarrow{AM} = \frac{1}{2}\overrightarrow{AB}$

$\Rightarrow m - a = \frac{1}{2}(b - a)$

$m = \frac{1}{2}b - \frac{1}{2}a + a = \frac{1}{2}b + \frac{1}{2}a = \frac{1}{2}(b + a)$

$m = \frac{1}{2}(a + b)$ (addition is commutative)

$m = \frac{1}{2}(\overrightarrow{OA} + \overrightarrow{OB})$

In general, given any line segment say $\overline{AB}$, the midpoint of the line AB is given by $m = \frac{1}{2}(a + b)$ where a and b are the position vectors of the points A and B respectively relative to the origin O.

Example 26

If P is the midpoint of QR where Q(-1, 4) and R(3, 6), find the position vector of P

Solution

By the mid-point theorem,

$p = \frac{1}{2}(q + r) = \frac{1}{2}\left\{\begin{pmatrix} -1 \\ 4 \end{pmatrix} + \begin{pmatrix} 3 \\ 6 \end{pmatrix}\right\} = \frac{1}{2}\begin{pmatrix} 2 \\ 10 \end{pmatrix} = \begin{pmatrix} 1 \\ 5 \end{pmatrix}$

Parallelogram

If ABCD is a parallelogram then in terms of vectors $\overrightarrow{AB} = \overrightarrow{DC}$ and $\overrightarrow{AD} = \overrightarrow{BC}$. Note the following special parallelograms.

1. A Square is a parallelogram with all the sides equal and all the angles being 90^0.
2. A Rectangle is a parallelogram with all the angles equal to 90^0.
3. A Rhombus is a parallelogram with all the sides equal but the angles are not right angle.

The above properties would be helpful when dealing with any of the above figures.

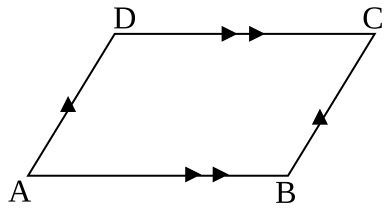


Fig. 4.7

Example 27

A parallelogram ABCD has vertices A(1, 3) B(-2, 5) and C(1, 8). Find the co-ordinates of the vertex D

Solution

The position vector of A, B and C are $a = \begin{pmatrix}1\\3\end{pmatrix}$, b = $\begin{pmatrix}-2\\5\end{pmatrix}$ and c = $\begin{pmatrix}1\\8\end{pmatrix}$ respectively.

Let (x, y) be the co-ordinates of D. ABCD is a parallelogram, it follows that $\overrightarrow{AB} = \overrightarrow{DC}$ and $\overrightarrow{AD} = \overrightarrow{BC}$

$\overrightarrow{AB} = b - a = \begin{pmatrix}-2\\5\end{pmatrix} - \begin{pmatrix}1\\3\end{pmatrix} = \begin{pmatrix}-3\\2\end{pmatrix}$

$\overrightarrow{DC} = c - d = \begin{pmatrix}1\\8\end{pmatrix} - \begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}1-x\\8-y\end{pmatrix}$

$\overrightarrow{AB} = \overrightarrow{DC} \Rightarrow \begin{pmatrix}-3\\2\end{pmatrix} = \begin{pmatrix}1-x\\8-y\end{pmatrix}$

From equality of vectors,

$\Rightarrow -3 = 1 - \text{x} \Rightarrow \text{x} = 4$ and

$2 = 8 - y \Rightarrow \text{y} = 6$

Therefore the co-ordinates of D are (4, 6)

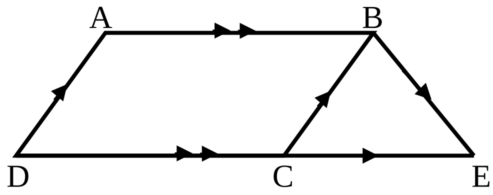


Fig. 4.8

Example 28

In the Figure 7.24, ABCD is a parallelogram. If $\overrightarrow{DA} = \begin{pmatrix}3\\5\end{pmatrix}$ *and* $\overrightarrow{CE} = \begin{pmatrix}2\\3\end{pmatrix}$, find $\overrightarrow{BE}$

Solution

From the figure, $\overrightarrow{BE} = \overrightarrow{BC} + \overrightarrow{\text{CE}}$ but $\overrightarrow{BC} = -\overrightarrow{DA}$

$\overrightarrow{BE} = -\overrightarrow{DA} + \overrightarrow{CE} = -\begin{pmatrix}3\\5\end{pmatrix} + \begin{pmatrix}2\\3\end{pmatrix} = \begin{pmatrix}-1\\-2\end{pmatrix}$

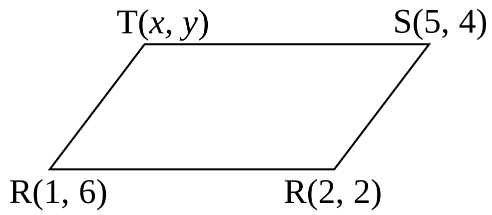
Example 29

The coordinates of the vertices of a parallelogram QRST are Q(1, 6), R(2, 2), S(5, 4) and T(x, y).

Find $\overrightarrow{QR}$ and $\overrightarrow{TS}$ and hence determine the values of x and y.

Calculate the magnitude of $\overrightarrow{RS}$

Solution



i. $\overrightarrow{QR} = r - q = \begin{pmatrix} 2 \\ 2 \end{pmatrix} - \begin{pmatrix} 1 \\ 6 \end{pmatrix} = \begin{pmatrix} 1 \\ -4 \end{pmatrix}$

$\overrightarrow{TS} = s - t = \begin{pmatrix} 5 \\ 4 \end{pmatrix} - \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 5 - x \\ 4 - y \end{pmatrix}$

$\overrightarrow{QR} = \overrightarrow{TS} \Rightarrow \begin{pmatrix} 1 \\ -4 \end{pmatrix} = \begin{pmatrix} 5 - x \\ 4 - y \end{pmatrix}$

$1 = 5 - x \Rightarrow x = 4$

$-4 = 4 - y \Rightarrow y = 8$

$x = 4, \;\; y = 8$

ii. $\overrightarrow{RS} = s - r = \begin{pmatrix} 5 \\ 4 \end{pmatrix} - \begin{pmatrix} 2 \\ 2 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$

$\left|\overrightarrow{RS}\right| = \sqrt{3^2 + 2^2} = \sqrt{13} units$

Example 30

If $p = \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \;\; q = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, r = \begin{pmatrix} 13 \\ 7 \end{pmatrix}$ and $kp + mq = r,$ where k and m

Solution

$kp + mq = r \Rightarrow k \begin{pmatrix} 2 \\ 1 \end{pmatrix} + m \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 13 \\ 7 \end{pmatrix} \Rightarrow \begin{pmatrix} 2k \\ k \end{pmatrix} + \begin{pmatrix} m \\ m \end{pmatrix} = \begin{pmatrix} 13 \\ 7 \end{pmatrix}$

$\begin{pmatrix} 2k + m \\ k + m \end{pmatrix} = \begin{pmatrix} 13 \\ 7 \end{pmatrix}$ Equating corresponding components

$\Rightarrow 2k + m = 13$ … … … (1) and

$k + m = 7$ … … … (2)

Solving equations (1) and (2) simultaneously,

$(1) - (2) \Rightarrow k = 6$

Substituting $k = 6$ into $(2), m = 1$

Therefore $k = 6$ and $m = 1$

Example 31

i. Given that $\overrightarrow{PQ} = \begin{pmatrix} -4 \\ 7 \end{pmatrix}$ and $\overrightarrow{RP} = \begin{pmatrix} 3 \\ -5 \end{pmatrix}$ evaluate $\overrightarrow{QR}$

ii. Describe precisely the relationship between $\overrightarrow{QR}$ and the vector $\overrightarrow{ZM} = \begin{pmatrix} -3 \\ 6 \end{pmatrix}$

Solution

From the vectors given, $\overrightarrow{QR} = \overrightarrow{QP} + \overrightarrow{PR}$

But $\overrightarrow{QP} = -\overrightarrow{PQ} = \begin{pmatrix} 4 \\ -7 \end{pmatrix}$ and $\overrightarrow{PR} = -\overrightarrow{RP} = \begin{pmatrix} -3 \\ 5 \end{pmatrix}$ Negative vectors

$\therefore \overrightarrow{QR} = \begin{pmatrix} 4 \\ -7 \end{pmatrix} + \begin{pmatrix} -3 \\ 5 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$

ii. $\overrightarrow{QR} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$ and $\overrightarrow{ZM} = \begin{pmatrix} -3 \\ 6 \end{pmatrix} \implies \overrightarrow{ZM} = -3\overrightarrow{QR}$

Hence $\overrightarrow{QR}$ is parallel to the vector $\overrightarrow{ZM}$ and are in opposite direction to each other.

Try these

A(9, 5) and B(3, 11) are points in the OXY plane. If C is the midpoint of AB, find
The value of the acute angle between $\overrightarrow{OC}$ and the x-axis
hey don't have any particular order

PRQ is an isosceles triangle in which PQ = QR and M is the midpoint of PR.
Show that $\overrightarrow{QP} + \overrightarrow{QR} = 2\overrightarrow{QM}$
If $\overrightarrow{PQ} = \begin{pmatrix} -3 \\ 4 \end{pmatrix}$ and $\overrightarrow{QR} = \begin{pmatrix} 4 \\ 3 \end{pmatrix}$

i. Express $\overrightarrow{QM}$ as a column vector

ii. Find $|\overrightarrow{QM}|$ correct to three significant figures.

If A(4, 7) is the vertex of a triangle ABC, $\overrightarrow{BA} = \begin{pmatrix} 5 \\ 3 \end{pmatrix}$ and $\overrightarrow{AC} = \begin{pmatrix} 4 \\ -3 \end{pmatrix}$,
Find the co-ordinates of B and C
If M is the midpoint of the line BC, find $\overrightarrow{AM}$
Given that $a = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$, $b = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$ and $c = \begin{pmatrix} 2 \\ 5 \end{pmatrix}$, evaluate

$3a - 4c$
$2b + 3c$

If $p = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$ and $q = \begin{pmatrix} -2 \\ 7 \end{pmatrix}$,
Find
i. $4p - 2q$
ii. $|4p - 2q|$

What can you say about your answer in $a(i)$ above and the vector $\begin{pmatrix} -1 \\ 8 \end{pmatrix}$?

Given $p = \begin{pmatrix} -7 \\ 6 \end{pmatrix}$, $q = \begin{pmatrix} 10 \\ -11 \end{pmatrix}$ and $k(p - q) = \frac{1}{2}\begin{pmatrix} -51 \\ 51 \end{pmatrix}$ find k

If $O(0,0)$, A(3, 1) and B(2, -1)are the co-ordinates of a quadrilateral OABC and if $\overrightarrow{BC} = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$, find the coordinates of C

ABCD is a quadrilateral, A, B and D have coordinates (0, 2), (2, 5) and (8, 0) respectively. If $\overrightarrow{AD} = 2\overrightarrow{BC}$, find the coordinates of C

P(-1, 2) and Q(x, y) are point on the Oxy plane such that $\overrightarrow{PQ} = \begin{pmatrix} 3 \\ -4 \end{pmatrix}$, find
The coordinates of Q
The bearing of Q from Q to the nearest degree.

XYZ is a triangle with vertices X(1, -3), Y(7, 5) and Z(-3, 5). L is the midpoint of the side XZ.
If O is the origin, express $\overrightarrow{XY}$, $\overrightarrow{YZ}$ and $\overrightarrow{ZX}$ as column vectors. Hence show that XYZ is isosceles.

Each class in the table consists of a single value.

Bearings

A bearing is used to represent the direction of one-point relative to another point. A bearing is measured in a clockwise direction from North. You might remember the cardinal points; North, South, East and West.

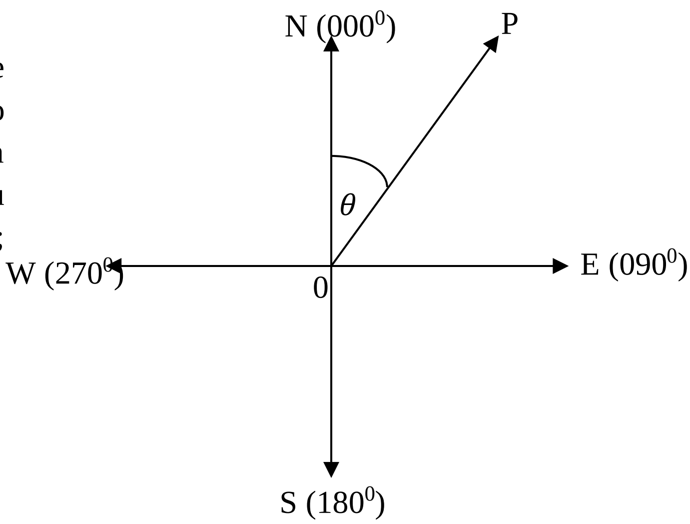


Fig. 4.9

Bearings are a way of describing the position of places/objects with respect to the cardinal points. They are read by using clockwise angles relative to the North Pole.

In bearing, we always use three digits, so a bearing of 4^0 is written as 004^0, a bearing of 65^0 is written as 065^0, a bearing of 215^0 is written as 215^0. Using bearing, a direction is written in terms of an angle θ, where $(000^0 \leq \theta \leq 360)$. Here, N, E, S and W are use to denote north, east, south and west respectively.

Fig. 5.11 gives an illustrations of the positions of A, B, C and D as bearing from O.

The bearing of A from O = $90^0 - 030^0 = 060^0$.
The bearing of B from O = $180^0 - 045^0 = 135^0$.
The bearing of C from O = $270^0 - 042^0 = 228^0$.
The bearing of D from O = $360^0 - 054^0 = 060^0$.

In general, if the bearing of B from A is θ, and is less than 180^0, $(i.e. 000^0 < \theta < 180^0)$, then the bearing of A from B is $(180 + \theta)^0$

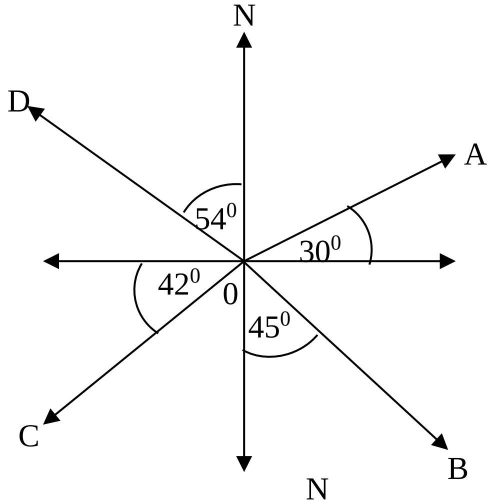


Fig. 4.10

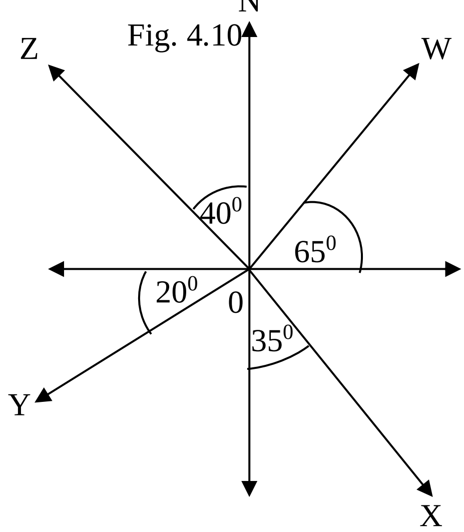


Fig. 4.11

Example 5

P, Q and R are three villages on a level ground. Q is 4km on the bearing 0400 from P, while R is 3km on the bearing 1300 from Q. Calculate the distance and bearing of P from R.

Solution

The problem is as illustrated in Fig 6.12.

ΔPQR is a right-angled triangle, with $\overrightarrow{PR}$ being the hypotenuse. By Pythagoras' theorem,

$|PR|^2 = |PQ|^2 + |QR|^2 \Rightarrow |PR|^2 = 4^2 + 3^2 = 16 + 9 = 25$

$|RQ| = \sqrt{25} = 5km$

The distance of P from R = 5km.

Let $\angle PRQ = \theta$

$tan\theta = \frac{|PQ|}{|QR|} = \frac{4}{3} = 1.33333,$

$\theta = \tan^{-1} 1.33333 = 53^0$ approximately.

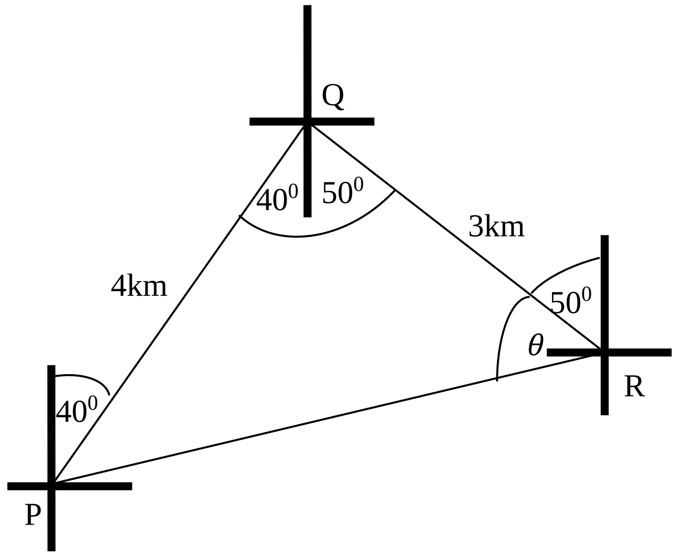


Fig. 4.12

Thus the bearing of P from R

= $3600 - \theta - 50^0$

$= 360 - 53 - 50 = 257^0$.

The bearing of R from P = $257^0 - 180^0 = 077^0$.

$\therefore \overrightarrow{PR} = (5km,\ 077^0)$

Example 6

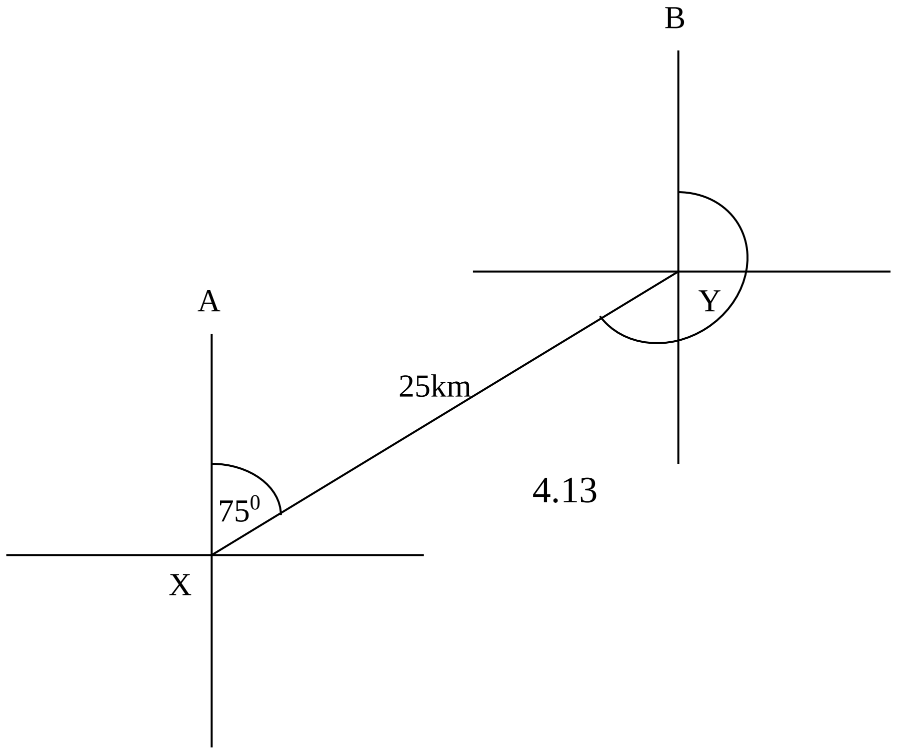
Y is 25 km from X on a bearing of 075^0. Determine the bearing to X from Y

Solution:

Draw a representative sketch. XA ∥ Y B since both lines are drawn to north. Therefore, $\angle AXY + \angle BYX$ add to 180^0. It follows that $\angle BYX = 105^0$.

The required bearing is the reflex angle $\angle BYX = 360^0 - 105^0 = 225^0$.

Drawing the axes in at the points is very useful in helping to find information.



4.13

Example 7

A ship sails from Port R on a bearing of 065^0 to Port S a distance of 54km. It then sails on a bearing of 155^0 from Port S to Port Q, a distance of 80km.

Find, correct to one decimal place:

The distance between R and Q

The bearing of Q from R.

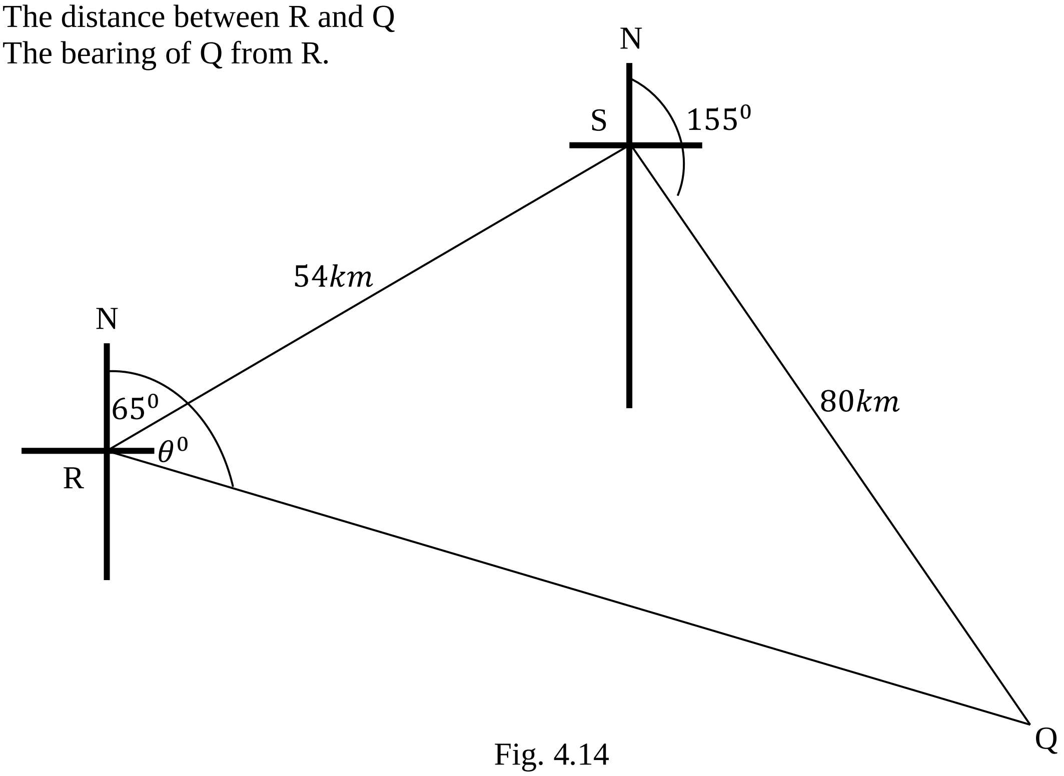


Fig. 4.14

Solution

a. Fig.6.13 shows the diagram for the problem. ΔRSQ is right-angled triangle with $\Delta RSQ = 90^0$.

By Pythagoras' theorem,

$|RQ|^2 = |RS|^2 + |SQ|^2 \Rightarrow |RQ|^2 = 54^2 + 80^2 = 9316$

$|RQ| = \sqrt{9316} = 96.52km$

b. Let $\angle SRQ = \theta$

$tan\theta = \frac{|SQ|}{|PS|} = \frac{80}{54} = 1.4815$ $\theta = \tan^{-1} 1.4815 = 55.98 \approx 56$

Thus the bearing of Q from R = $065^0 + 056^0 = 121^0$.

Try these

P, *Q* and *R* are three points on level ground with *P* due north of *R*. Angle *QPR* = 40º and *PQ* = *PR*. Calculate the bearing of

(a) *Q* from *P*, **(b)** *P* from *Q*, **(c)** *Q* from *R*, **(d)** *R* from *P*.

2) Points A, B, C and D lie on level ground. The point D is due north of A. $\angle DAC = 140^0$, $\angle CAB = 90^0$ and $\angle ABC = 75^0$.

Find the bearing of

(a) *A* from *C*, **(b)** *A* from *B*, **(c)** *C* from *A*, **(d)** *C* from *B*

3) Jeffery walks 3 km due east from a point P to Q. From Q, he walks a further distance of 5 km on a bearing of 054o to a point R. Calculate the distance of PR and find the bearing of R from P.

4) Two ships leave a port at the same time. One sails at 22 km/h on a bearing of 047o and the other at 18 km/h on a bearing of 148o . Find the distance between the two ships after 4 hours.

5) The points P, Q and R are in the same plane. $\overrightarrow{PQ} = (x\ km,\ 030^0)$, $\overrightarrow{RP} = (12\ km,\ 300^0)$ and $|QR| = 20km.$

a. Find x

b. If the point S is on $\overrightarrow{PR}$ such that $\angle PQS = 45^0$, find $|QS|$ in the form $p\sqrt{q}$, where p and q are integers.

Angle of Elevation and Depression

When a person looks at something above his or her location, the angle between the line of sight and the horizontal is called the **angle of elevation**. In this case, the line of sight is "elevated" above the horizontal.

When a person looks at something below his or her location, the angle between the line of sight and the horizontal is called the **angle of depression**. In this case, the line of sight is "depressed" below the horizontal.

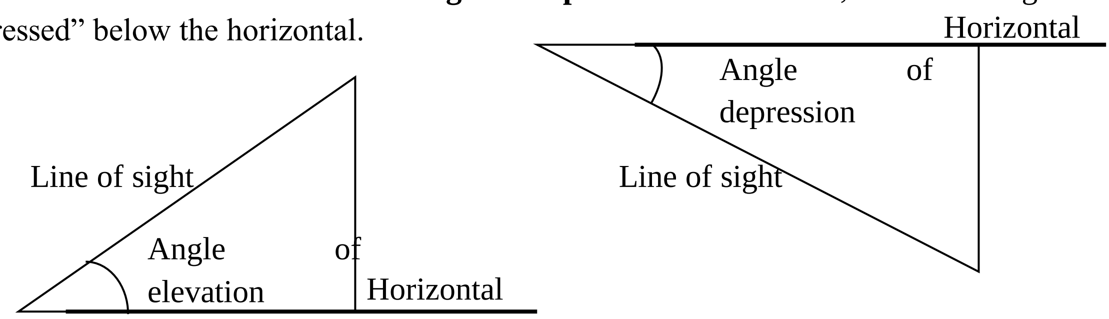


Fig. 4.15

Example 8

A person stands at the window of a building so that his eyes are 12.6 m above the level ground in the vicinity of the building. An object is 58.5 m away from the building on a line directly beneath the person. Compute the angle of depression of the person's line of sight to the object on the ground.

Solution:

The angle of depression of the line of sight is the angle, θ, that the line of sight makes with the horizontal, as shown in the Fig. 4.6. Since the ground is level, it is parallel to any horizontal line, and so the angle that the line of sight makes with the ground is equal to θ as well

As a result,

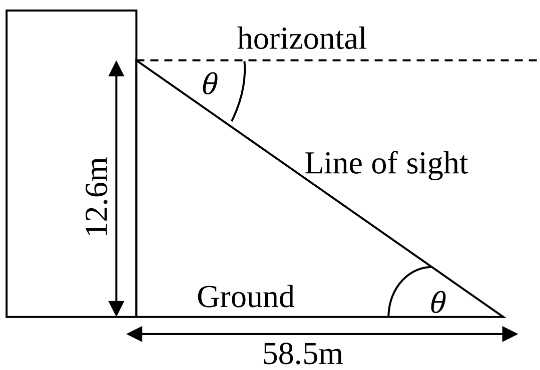


Fig. 4.16

$tan\theta = \frac{12.6}{58.5}$

$\theta = \tan^{-1}\frac{12.6}{58.5} = 12.15^0$

The angle of depression of the line of sight to the object is 12.150 rounded to two decimal places.

Example 9

Calculate the angle of elevation of the line of sight of a person whose eye is 1.7 m above the ground, and is looking at the top of a tree which is 27.5 m away on level ground and 18.6 m high.

Solution

The angle of elevation is the angle the line of sight makes with the horizontal when the line of sight is upwards or above the horizontal (in contrast to the situation where we use the term "angle of depression" to refer to a line of sight which is downwards, or below the horizontal). So, schematically, the situation here is as shown in the figure to the right, with the symbol θ indicating the required angle of elevation.

Note that the right triangle for which the line of sight forms the hypotenuse is 16.9 m high after we take into account the 1.7 m distance that the observer's eye is above the ground.

Thus

$$tan\theta = \frac{16.9}{27.5}$$

so that

$\theta = \tan^{-1}\left(\frac{16.9}{27.5}\right) \cong 31.57^0$

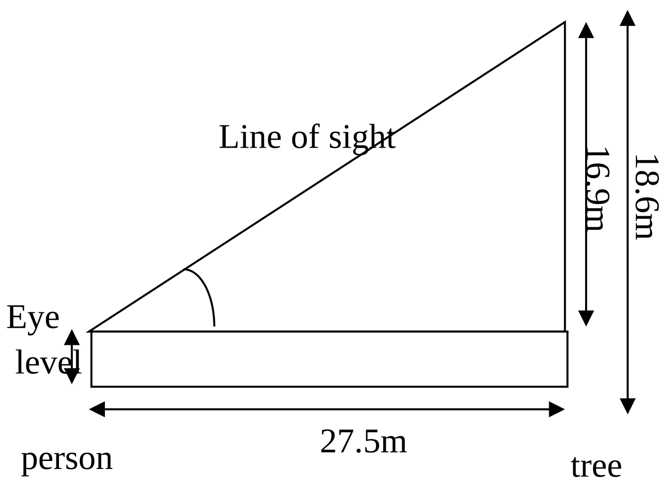


Fig. 4.17

Thus, to two decimal places, the angle of elevation of this person's line of sight is 31.57^0.

Try these

1) A campsite is 9.41 miles from a point directly below the mountain top. If the angle of elevation is 12º from the camp to the top of the mountain, how high is the mountain?

2) How far from the door must a ramp begin in order to rise three feet with an 8º angle of elevation?

1) An A-frame cabin is 26.23 feet high at the center, and the angle the roof makes with the base is 53º15'. How wide is the base?

2) The side of a lake has a uniform angle of elevation of 15º 30'. How far up the side of the lake does the water rise if, during the flood season, the height of the lake increases by 7.3 feet?

3) A building casts a shadow of 110 feet. If the angle of elevation from that point to the top of the building is 29º 3', find the height of the building.

4) From a point 120 feet from the base of a church, the angles of elevation of the top of the building and the top of a cross on the building are 38º and 43º respectively. Find the height to the top of the cross. (The ground is flat.)

5) From the top of a fence, a person sites a lion on the ground at an angle of depression of 24^0 . If the man and the fence is 4.2 meters high, how far is the man from the lion?

STRAND 5 Mensuration: ***Learning, teaching and applying***

Area of a rectangle

The measure of the amount of surface occupied by an object is termed as it' Area. Area is therefore the measurement of the amount of surface occupy by an object. It is a two-dimensional measure.

A Unit square, preferably square centimeter is used to cover the surface of the rectangular figure.

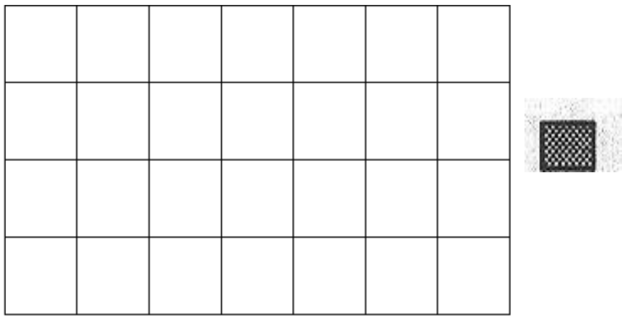


Fig. 5.1

In the diagram, supposing each box is a unit square, then the area of the rectangle will be how many of the small boxes we have in the rectangle, by counting the column, we have 7 boxes and the rows give 4, instead of counting from beginning to end, we could just multiply the number on row by column which is 7 x 4 = 28

This is just like multiplying the how long the length (L) is by how long the breadth (B) is. Therefore, the area of a rectangle is given as

A = L x B

Example 1

Find the area of a rectangle whose length is 8cm and breadth is 4cm

Solution

Area A = L x B

= 8cm x 4cm

= $32cm^2$

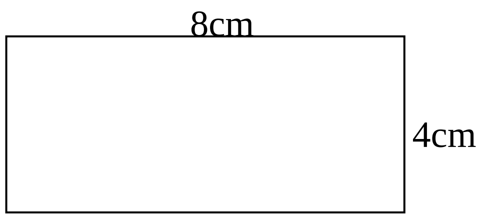


Fig. 5.2

Example 2

A rectangle has an area of $171cm^2$. If the length is 19cm, find the how long the breadth is.

Solution

Area A of the rectangle = L x B

$171cm^2$ = 19cm x B

$$B = \frac{171cm^2}{19cm} = 9cm$$

Example 3

Find the area of the Fig. 5.3

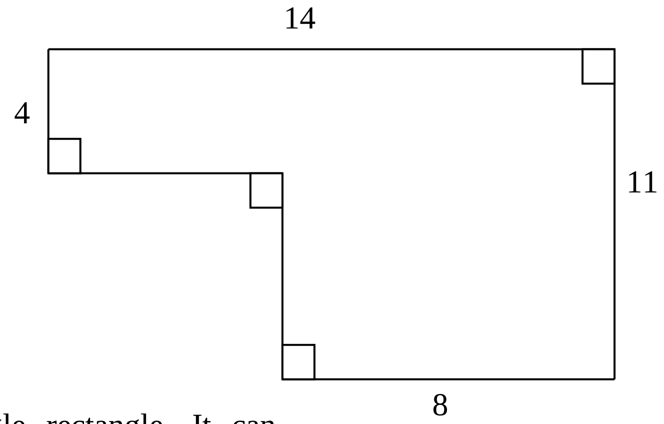


Fig. 4.3

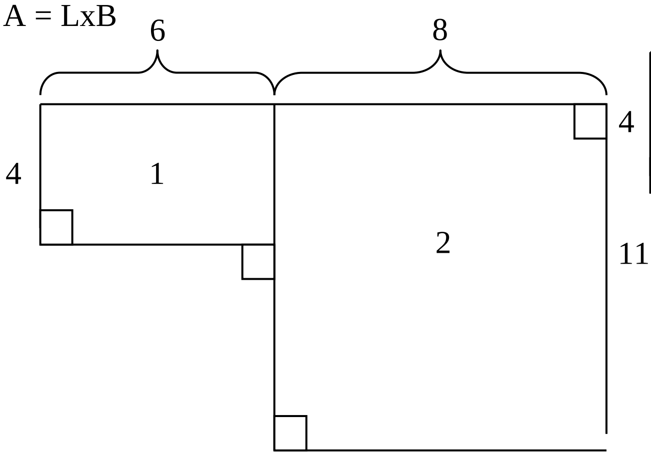
Solution:

This figure is not a single rectangle. It can, however, be broken up into two rectangles. We then will need to find the area of each of the rectangles and add them together to calculate the area of the whole figure.

There is more than one way to break this figure into rectangles. We will only illustrate one below.

we know that, the trea of a Rectangle

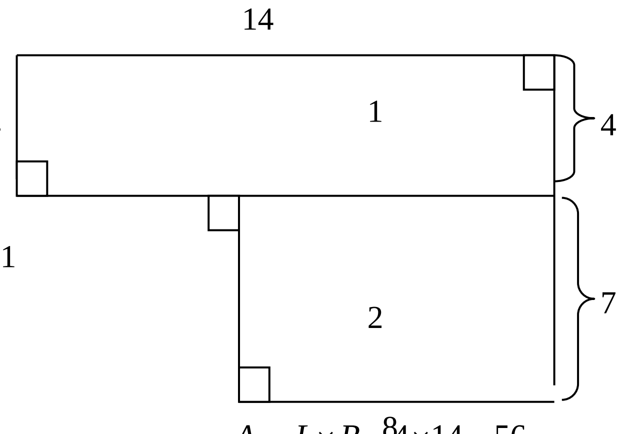
A = LxB



$A_1 = L \times B = 4 \times 6 = 24$

$A_2 = L \times B = 8 \times 11 = 88$

$A_1 + A_2 = 24 + 88 = 112 units$



$A_1 = L \times B = 4 \times 14 = 56$

$A_2 = L \times B = 8 \times 7 = 56$

$A_1 + A_2 = 56 + 56 = 112 units$

Example 4

A rectangular park of length 30 m and breadth 24 m is surrounded by a 4 m wide path. Find the area of the path.

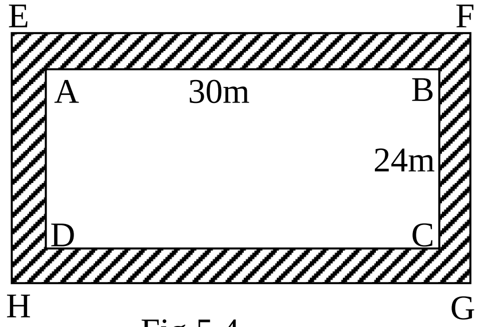


Fig.5.4

Solution:

Let ABCD be the park and shaded portion is the path surrounding it (See Fig. 4.16).

So, length of rectangle EFGH = (30 + 4 + 4) m = 38 m, Breadth of rectangle EFGH = (24 + 4 + 4) m = 32 m

Therefore, area of the path = area of rectangle EFGH – area of rectangle ABCD

= (38 × 32 – 30 × 24) m2

= (1216 – 720) m^2

= 496 m^2

Try these

1. The length of a diagonal a rectangle is 13cm. If its length is 12cm, find the area of the rectangle.

Find the area of the each of the following

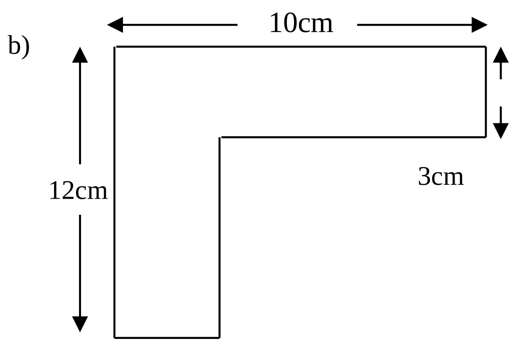


Fig. 5.6

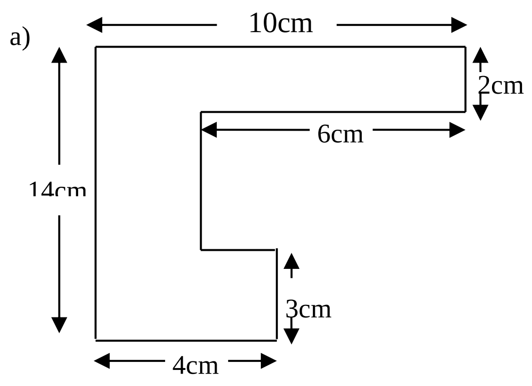


Fig. 5.7

2. A square 7cm is removed from a rectangular sheet of dimension 9cm by 12cm. Find the area of the remaining portion.
3. The length of a rectangular sheet of metal is twice its breadth. If the perimeter of the rectangle is 18cm, find the area of the rectangle.
4. The perimeter of a rectangle is 36cm. If the area is 80cm^2, determine the length and breadth of the rectangle.
5. A rectangular sheet of metal, of negligible thickness, made of uniform material is 14cm long and 10 cm wide. Six square holes of dimension ***r***cm are drilled through the sheet.
6. Find an expression, in terms of ***r***, for the area of the metal left after the drilling.
7. If the ratio of the new weight of the sheet to the original weight is 11:35, find the value of ***r***.

The Area of a Triangle

To find the area of a triangle, we apply the concept of the area of a rectangle. Let us first consider the area of a right-angled triangle

Draw a rectangular shape of given length (L) and breath (W), cut it out.

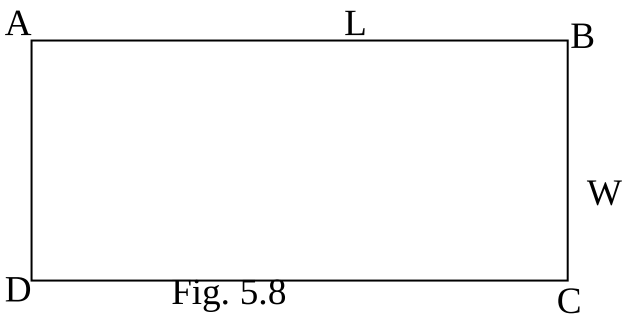


Fig. 5.8

(i)

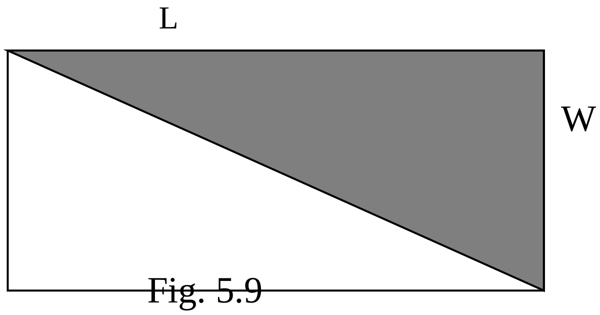


Fig. 5.9

(ii)

Cut along of its diagonal say BD to obtain two right angle triangles. When we compare the two right angled triangles, we will discover that the two areas are equal. This means the area of the right angled triangle = ½ of the area of the rectangle = ½ L x W

Comparing the right-angled triangle formed from the rectangle we will see that the width of the rectangle is the height of the triangle whilst the length becomes the base of the triangle

Area of the right angled triangle = ½bh.

To find the area of any triangle, make another rectangle cut out with length L and width W. Mark a point P on the side of the rectangle

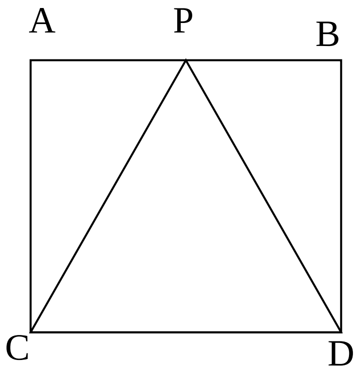


Fig.5.10

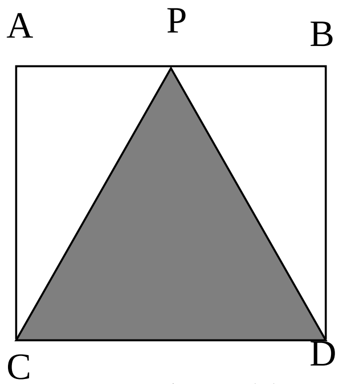


Fig. 5.11

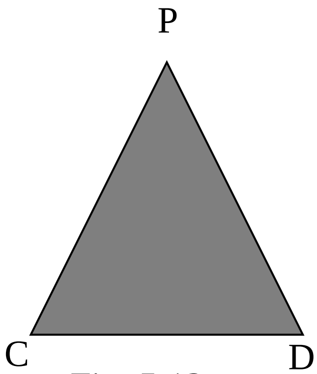


Fig. 5.12

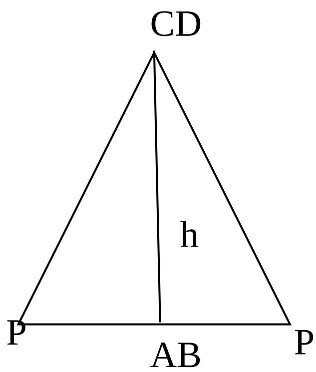


Fig.5.13

Join the point P to the opposite corners of the rectangle, Shade the triangle formed within the figure 5.10 Cut out the shaded triangle and join the two unshaded triangles together to form one triangle.

When we compare the shaded triangle with the unshaded triangle, we will discover that the unshaded triangle fit exactly on the shaded triangle and hence will have the same area. As the triangle stands, the width becomes a height whilst the length becomes a base. The area of the triangle will therefore be equal to half the area of the rectangle. Area of triangle = ½ bh.

Example 6

Find the area of the triangle whose base is 8cm and height 12 cm.

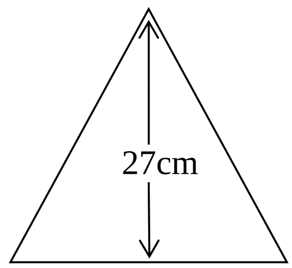
Solution

Area of the triangle $=\frac{1}{2}b\times h=\frac{1}{2}\times 8\times 12=48cm$

Example 7

The area of a triangle is 108cm^2 and the height is 27cm, find its base.

Solution



4.25

$$A=\frac{1}{2}\times b\times h$$

$$b=\frac{2A}{h}=\frac{2\times 108}{28}=7.71cm \approx 8cm$$

or

$$A=\frac{1}{2}\times b\times h$$

$$108=\frac{1}{2}\times b\times 28 \Rightarrow 108=b\times 14$$

$$b=\frac{108}{14}=7.71cm \approx 8cm$$

Area of a Parallelogram.

Our next formula will be for the area of a parallelogram. A parallelogram is a quadrilateral with opposite sides parallel. It therefore follows that, a rectangle, which has four right angles, with the opposite sides being equal and parallel is an example of a parallelogram.

THEOREM 1

If both pairs of opposite sides of a quadrilateral are congruent and parallel, then the quadrilateral is a parallelogram.

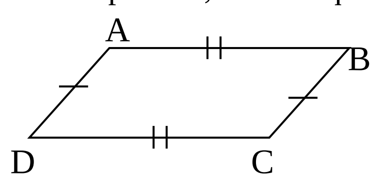


Fig 5.14

THEOREM 2

If both pairs of opposite angles of a quadrilateral are congruent, then the quadrilateral is a parallelogram.

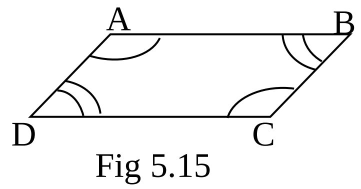


Fig 5.15

THEOREM 3

If an angle of a quadrilateral is supplementary to both of its consecutive angles, then the quadrilateral is a parallelogram.

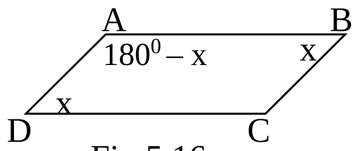


Fig 5.16

THEOREM 4

If the diagonals of a quadrilateral bisect each other, then the quadrilateral is a parallelogram.

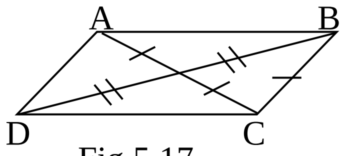


Fig 5.17

The parallelogram ABCD in Fig. 5.17 below is composite figure of two congruent triangles, that is $\Delta AEF\ and\ \Delta DBC$ and a rectangle ABDE.

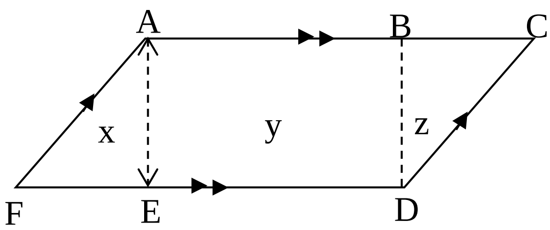


Fig 5.18

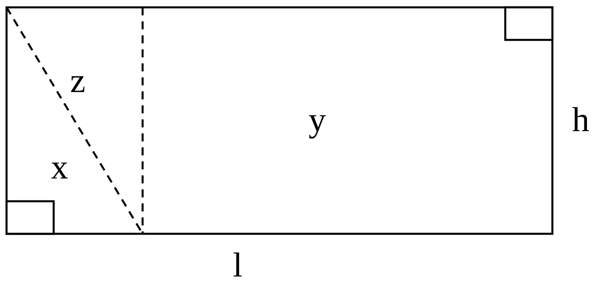


Fig 5.19

Cut triangle DBC and join it at the other side like we have in Fig 4.31 to get a rectangle whose area is $l \times b$. Hence

The area of a parallelogram ACDF is given by the length of the base (l) x the perpendicular height (h)

$$\boldsymbol{Area\ of\ a\ parallelogram = l \times h}$$

Example 8

Fig. 4.32 shows a field ABCD in a parallelogram shape. Calculate

- The length of AT
- The area of the field.

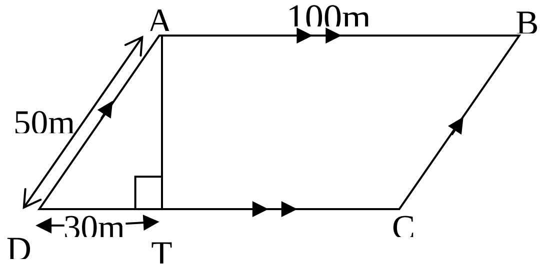


Fig. 5.20

Solution

a) ΔATD is a right-angled triangle with AB being the hypotenuse. By Pythagoras' theorem,

$|AD|^2 = |DT|^2 + |AT|^2$

$|AT|^2 = |AD|^2 - |DT|^2 = 50^2 - 30^2 = 1600$

$|AT| = \sqrt{1600} = 40$

b) The length of the base (l) = 100m

The height of the parallelogram (h) = 40m.

Area of the parallelogram = l $\times$ h

$= 100\text{m} \times 40\text{m} = 4000m^2$

Example 9

Find the area of the shaded portion in Fig. 4.33

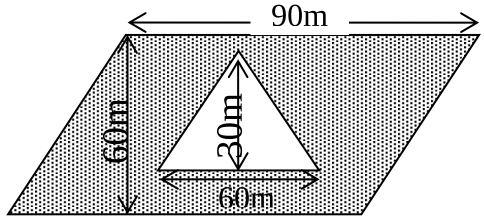


Fig. 5.21

Solution

Area of the triangle $= \frac{1}{2} \times b \times h$

$= \frac{1}{2} \times 60m \times 30m = 900m^2$

Area of parallelogram $= l \times h$

$= 90m \times 60m = 5400m^2$

Area of the shaded portion = Area of Parallelogram – Area of triangle

$= 5400m^2 - 900m^2 = 4500m^2$

Area of a Rhombus

Area of a rhombus is also the base x the perpendicular height. This is because a rhombus is a parallelogram with all four sides equal. Or

Considering Fig.4.34, ABCD is a rhombus. The diagonals AC and BD bisect each other at right angle. That is $|AT| = |TC|$ and $|BT| = |TD|$ and $\angle DTC = 90^0$

When you divided the rhombus into two congruent isosceles triangles, i.e. ΔACD and ΔACB:

The height of $\Delta ACD = |DT| = \frac{1}{2}|DB|$

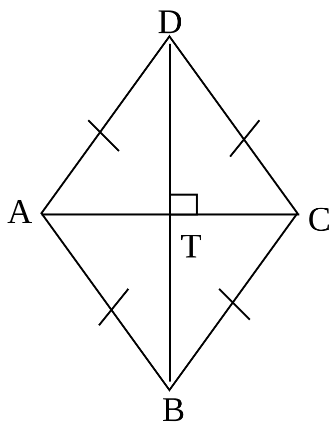


Fig. 5.22

The area of $\Delta ACD = \frac{1}{2} \times base \times height = \frac{1}{2} \times |AC| \times \frac{1}{2}|DB| = \frac{1}{4} \times |AC||DB|$

The two triangles are congruent, therefore the area of $\Delta ACD\ = the\ area\ of\ \ \Delta ACB$

The area of the rhombus = $\Delta ACD\ +\ \ \Delta ACB = 2 \times \left(\frac{1}{4} \times |AC||DB|\right) = \frac{1}{2}|AC||DB|$

$$The\ area\ of\ the\ rhombus\ \ = \frac{1}{2} \times\ the\ product\ of\ the\ diagonals.$$

Example 10

If the length of the diagonals of a rhombus are 24cm and 10cm, find:

i. the area of the rhombus,

ii. the perimeter of the rhombus

Solution

i. The area of the rhombus = $\frac{1}{2}\ \times\ the\ product\ of\ the\ diagonals.$

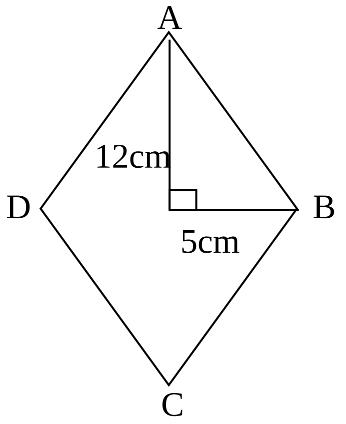
$$= \frac{1}{2} \times 24 \times 10 = 120cm^2$$

ii. ΔATB is a right angled triangle with AB being the hypotenuse. By Pythagoras' theorem,

$$|AB|^2 = |AT|^2 + |BT|^2 = 12^2 + 5^2 = 169$$

$\Longrightarrow |AB| = \sqrt{169} = 13cm$

This means that each side of the rhombus is 13cm. The perimeter therefore $= |AB\,| + \ |BC|\ +\ |CD|\ +\ |DA| = 4 \times 13 = 52cm$



Area of a Trapezium

Area of $\Delta AND = \frac{1}{2} \times |AN| \times h$

Area of $\Delta CMB = \frac{1}{2} \times |MB| \times h$

Area of rectangle DCMN = $|NM| \times h$

Total Area = $\frac{1}{2} \times |AN| \times h + |NM| \times h + \frac{1}{2} \times |MB$

$$= \frac{1}{2}h(|AN| + 2|NM| + |MB|)$$

But $|AN| + |NM| + |MB| = a$

$\Longrightarrow Total\ Area = \frac{1}{2}h(a + |NM|)$

Again $|NM| = |DC| = b$

$\therefore Total\ Area = \frac{1}{2}h(a + b)$

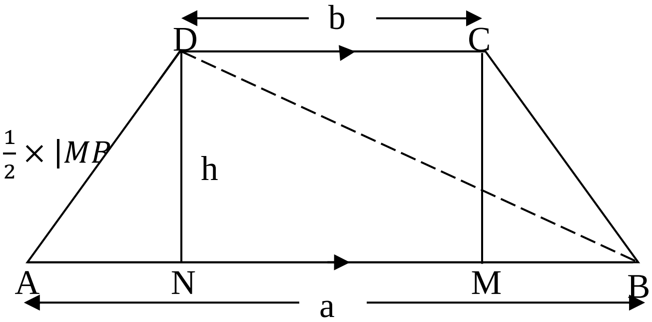
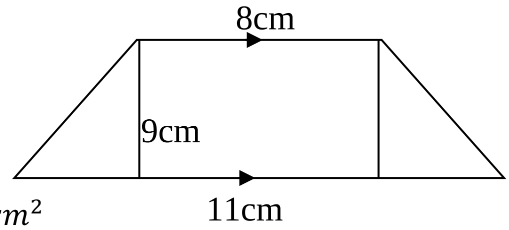


Fig 5.23

Example 11

The parallel sides of a trapezium are 11cm and 8cm. If the distance between the parallel lines is 9cm, find the area of the trapezium.



Solution

$a = 11cm, \ \ b = 8cm \quad and \quad h = 9cm$

$Area\ of\ a\ trapezium = \frac{1}{2}h(a+b)$

$= \frac{1}{2} \times 9cm(11cm + 8cm) = \left(\frac{1}{2} \times 9 \times 19\right) cm^2 = 85.5cm^2$

Example 12

Length of the two parallel sides of a trapezium are 20 cm and 12 cm and the distance between them is 5 cm. Find the area of the trapezium

Solution:

$Area\ of\ a\ trapezium = \frac{1}{2}h(a+b)$

$= \frac{1}{2} \times 5(20 + 12) = \frac{1}{2} \times 5 \times 32 = 80cm^2$

Try these

1. Length and breadth of a rectangular field are 22.5 m and 12.5 m respectively. Find:

(i) Area of the field (ii) Length of the barbed wire required to fence the field

1. The length and breadth of rectangle are in the ratio 3 : 2. If the area of the rectangle is 726 m2, find its perimeter.
2. Find the area of a parallelogram whose base and corresponding altitude are respectively 20 cm and 12 cm.
3. Area of a triangle is 280 cm2. If base of the triangle is 70 cm, find its corresponding altitude.
4. Find the area of a trapezium, the distance between whose parallel sides of lengths 26cm and 12 cm is 10 cm.
5. The Perimeter of a rhombus is 146 cm and the length of one of its diagonals is 48 cm. Find the length of its other diagonal.
6. The perimeter of a rhombus is 40cm. If one of its diagonal is 16cm, find the area of the rhombus.
7. Two parallel sides of a trapezium are 60cm and 77cm and the other sides are 25cm and 26cm. find the area of the trapezium
8. If the sides of a rhombus are each 5m long and length of one of the diagonals is 8m long, find the length of the other diagonal

9. If the diagonals of a rhombus are 8cm and 6cm long, find:

i. The area of the rhombus
ii. The length of the sides of the rhombus

10. The acute angle between the sides.

Find the area of the figure ABCDEFG (See Fig. 4.36 in which ABCG is a rectangle, AB = 3 cm, BC = 5 cm, GF = 2.5 cm = DE = CF., CD = 3.5 cm, EF = 4.5cm, and CD || EF.

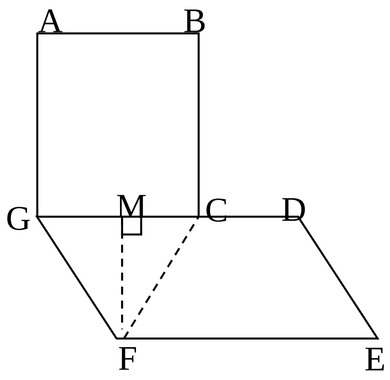


Fig. 5.24

Area of Solid Figures

Solid figure is a three dimensional (3-D) figure (length, width and height); 2nd grade solid figures include prisms, pyramids, cylinders, cones, and spheres. Measuring the surface (or boundary) constituting the solid. It is called the **surface area** of the solid figure.

CUBOIDS AND CUBES

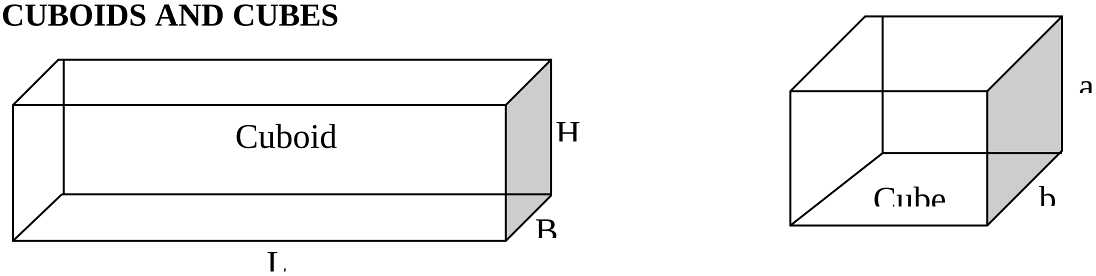


Fig. 5.25

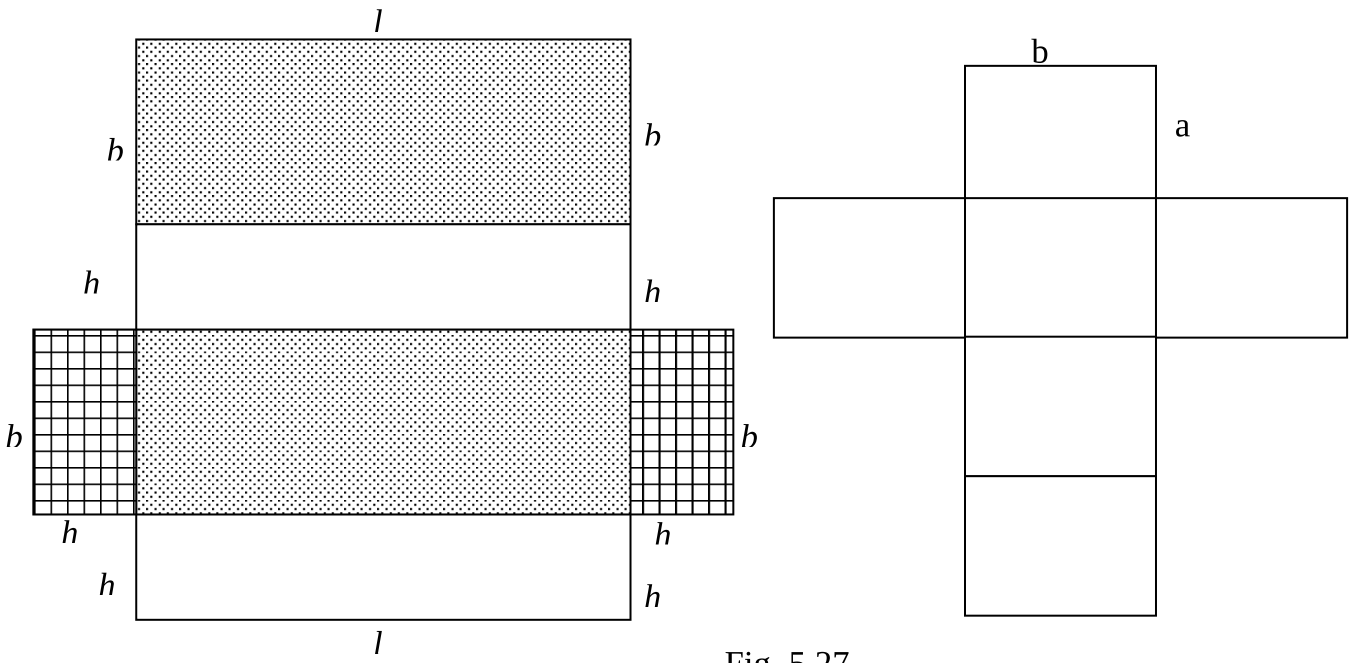


Fig. 5.27

We can see that the cuboid is made up of three pairs of congruent faces. The dimension of the pairs are (l by b), (l by h) and (b by h). Therefore the total surface area S of the cuboid is $2(l \times b) + 2(l \times h) + 2(b \times h)$

$S = 2\{(l \times b) + (l \times h) + (b \times h)\}$

$volume\ V = l \times b \times h$

Example 13

The toolbox with a lid has dimensions 22cm by 30cm by 16cm. Find the total surface area of the box.

Solution

The total surface are $= 2\{(22 \times 30) + (22 \times 16) + (30 \times 16)\}cm^2$

$= 2(660 + 352 + 480)cm^2 = 2984cm^2$

Example 14

Fig. 5.28 is a model church auditorium in Accra College of Education. The model is in the shape of a cuboid with a triangular prism on top. Find

i. the cross sectional area.

ii. the volume of the model church.

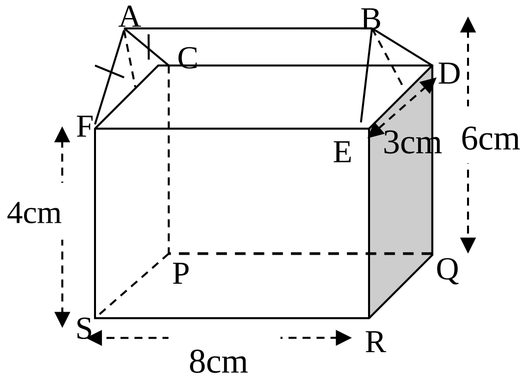


Fig 5.28

Solution

The cross sectional is a composite figure consisting of a triangle BDE and a rectangle DQRE.

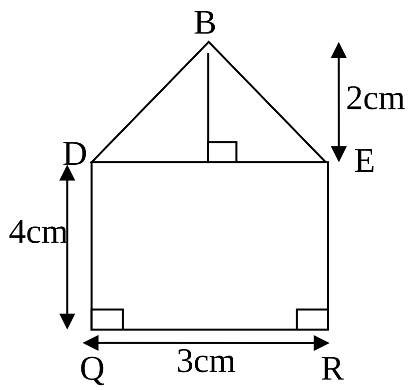
i. The base of the triangle = 3cm, The height of the triangle = 2cm

Area of triangle = ½ x base x height $= \frac{1}{2} \times 3 \times 2 = 3cm^2$

The length of the rectangle = 4cm, The breadth = 3cm

Area of the rectangle = length x breadth $= 4 \times 3 = 12cm^2$

Hence the cross sectional area = $3cm^2 + 12cm^2 = 15cm^2$



ii. The length of the model church is 8cm,

The volume = cross sectional area x length of the model

$= 15cm^2 \times 8cm = 120cm^3$

CIRCLE

A circle is a plane figure bounded by a curved line called the CIRCUMFERENCE. Any point on the circumference is equidistant from a fixed point called the CENTER.

Semi circle : this a half of a circle.

The other parts of a circle are illustrated and described in Fig. 3.1.

Chord: This is a straight line joining any two points on the circumference. The chord divides the circle into segments. The bisector of the chord passes through the center of the circle.

Radius: This is a straight line drawn from the centre of the circle to any point on the circumference. Plural is radii.

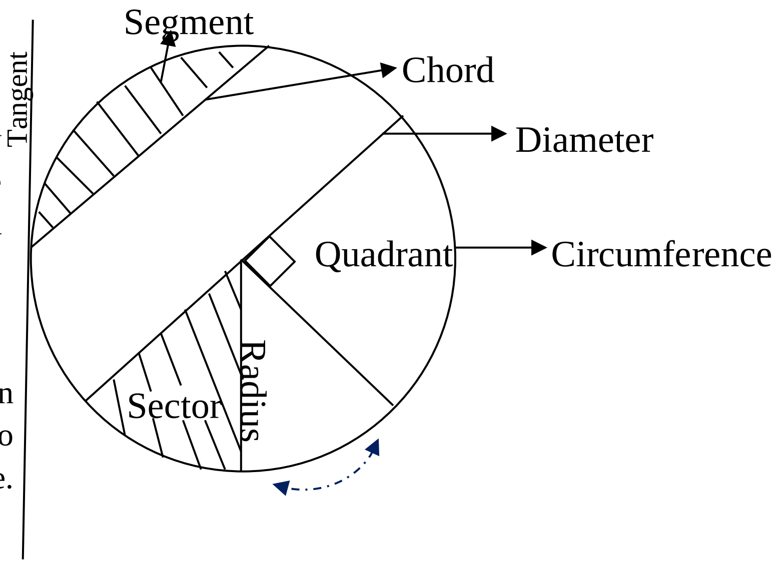


Fig. 5.30

Diameter: It is a special chord that pass through the centre of the circle. It is twice the radius in length. It is the axes of symmetry and divides the circle in to semi circles.

Secant: The secant is a line which does not pass through the centre of a circle and which intersects the circumference and splits the circle into two regions.

Arcs: The arc is the circumference of any incomplete circle.

Tangent: This is a straight line which TOUCHES but does NOT CUT the circumference or arc. The tangent touches the circumference at one and only one point called the point of contact.

Congruency: Two circles with the same radius but different centres are said to be congruent. In particular, they have the same area and can be completely superimposed on each other.

Segment: This is the area bounded by a chord and an arc that cuts it off.

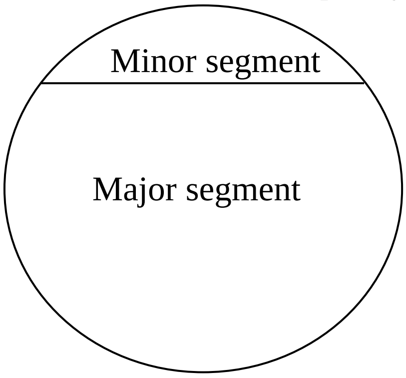


Fig5.31

A chord divides the circle into two,

a) The Minor (smaller) segment
b) The Major (bigger or larger) segment

Concentric Circles: Two circles with the same centre but different radius are said to be concentric. The diameter of the larger circle contains the corresponding diameter of the smallest circle. The line joining the centres of the two circles of different radius is an axis of symmetry for both circles.

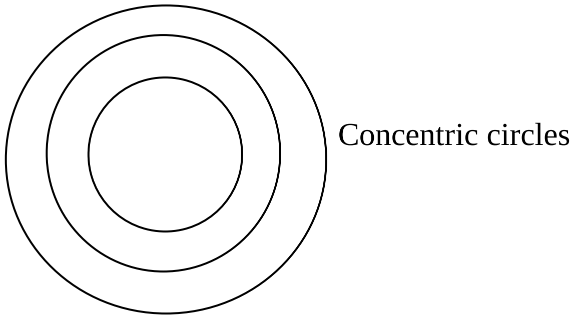


Fig. 5.32

Sector: A sector is the area bounded by two radii of a circle and the arc they cut off i.e. Minor and Major sectors.

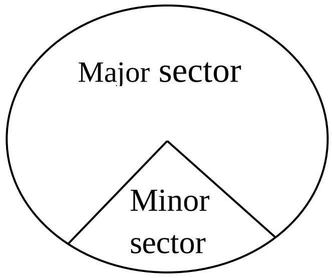


Fig. 3.33

Quadrant: This is the area bounded by two radii of a circle which are at right angles to each other and the arc they cut off. It is a quarter of the total area of a circle.

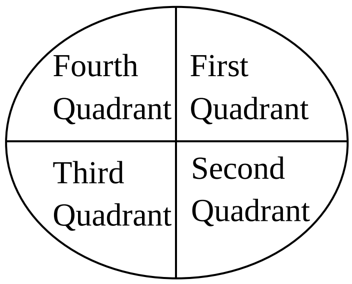


Fig. 5.34

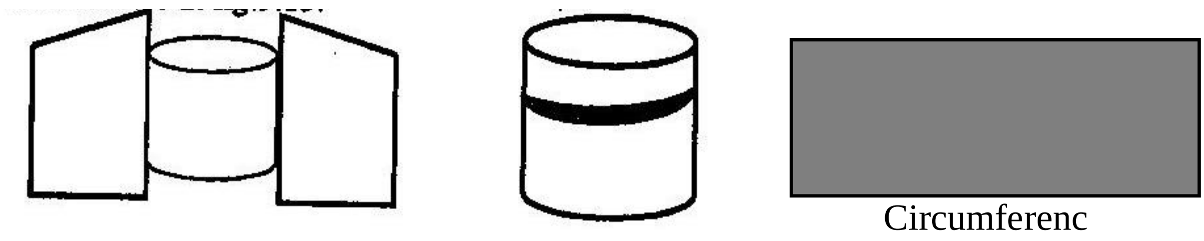
Pi (π): For any circle, the ratio of the circumference to the diameter (also the area to the square of the radius is a constant, and this constant is called Pi.

Circumference of a Circle

To determine the circumference of a circle, try to perform activities with circular objects like milk tin, milo tin, torch battery etc.

- measure the diameter of each of the circular objects. The measurement of the diameter is done by placing the circular object length in between two cards of the same lengths and record.

- measure the circumference (perimeter) of each of the circular objects. The measurement of the circumference is done by placing a length of string, rope or thread around the object, cut and measure on a measuring instrument and record.



measuring diameter and circumference

Diameter and circumference

Object	Circumference	Diameter	C/d
Milk tin	21.4	6.9	3.147
Milo tin	81.4	10.0	3.140
Torch battery	10.7	3.4	3.147
Nido tin	32.3	10.3	3.127
Geisha tin	17.2	5.5	3.14

The ratios of the circumference to the diameter of each of the circular objects are calculated and the results recorded in a table. Discover that in each circular object the ratio c/d are approximately the same and that, this ratio is constant. We call this constant pi denoted by π. π therefore has an approximate value of 3.14 which is about 22/7

The circumference, the distance round the circle C is therefore given as

$\frac{C}{D} = \pi$, where C is the circumference, D is the diameter and π is the constant 3.14. making C the subject, C = πD. Remembering that two radii makes a diameter, replace D with 2r

∴ the circumference of a circle is given by C = 2πr or πd

C = π x 2r C = 2π r

Example 1

Find the circumference of a circle with radius 14cm. $\left(Take\ \pi = \frac{22}{7}\right)$

Solution

The circumference C is given by

$C = 2\pi r$, and $\pi = \frac{22}{7}$, $r = 14$

$C = 2 \times \frac{22}{7} \times 14$ = 88cm.

Example 2

The circumference of a circle is 132cm. find the radius of the circle if $\pi = \frac{22}{7}$

Solution

The circumference C of the circle is given by

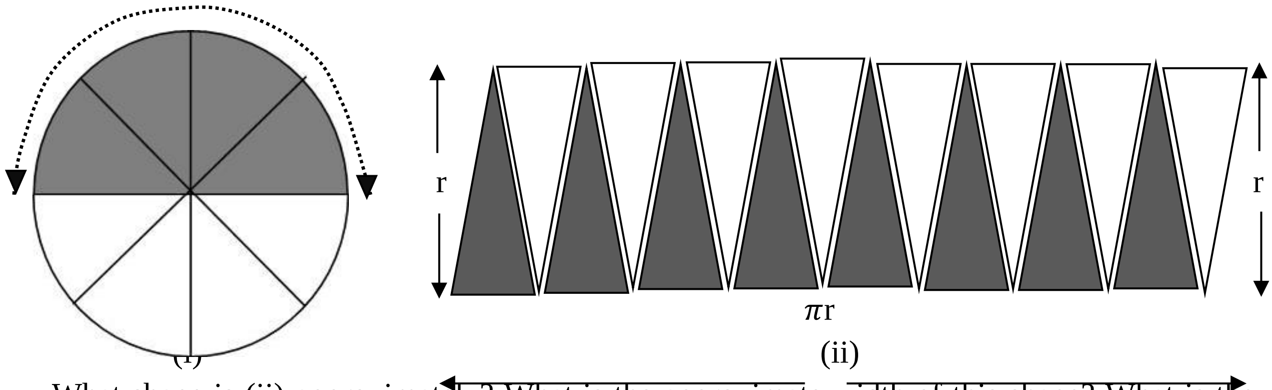
$C = 2\pi r$, and $c = 132, \pi = \frac{22}{7}$,

$\therefore 132 = 2 \times \frac{22}{7} \times r$

$r = \frac{132}{2} \times \frac{7}{22} = 21$

Area of a Circle

Draw a circle with radius of about 4cm on a card. Cut out the circle to obtain a circular cut out. Fold the circular cutout along a diameter into halves and then colour one half. Fold again to form quarters and again and again to eighths and sixteenths, cut the sectors out and arrange to form a rectangle as shown below;



What shape is (ii) approximately? What is the approximate width of this shape? What is the length of the shape?

We note that the curved edge of each semi-circle is half the circumference and is therefore of length ½ x 2π r = πr

When we cut along the folded lines we will obtain 8 coloured sectors and 8 uncoloured sectors. Arrange the sectors together to form a rectangle (ii) This shape is approximately a rectangle with length πr and breadth r. giving the area of a circle as:

Area = $\pi\ r\ x\ \ r\ \ \therefore A = \ \pi r^2$

Since all the sectors in (i) are arranged in (ii), the area of the circle is equal to the area of the shape in (ii). If the number of sectors is large, the shape in (ii) will look more like a rectangle. Its length will then be half of the circumference of the circle. The width will then be equal to the radius, r, of the circle.

Therefore the area of circle A = area of rectangle = $\pi r \times r = \pi r^2$ as shown earlier.

This can also be written as

$A = \pi r \times r \times \frac{4\pi}{4\pi}$ (Any number divided by itself = 1 and also one times any number is the same number.

$A = \pi r \times r \times \frac{2 \times 2\pi}{4\pi} = \frac{2\ \pi r \times 2\pi r}{4\pi}$, we learnt that $C = 2\pi r$

We can conclude that $A = \frac{C \times C}{4\pi} = \frac{C^2}{4\pi}$

Also from $A = \pi r^2$ we can say that $A = \pi \times r \times r = \pi r \times r \times \frac{4}{4}$

$A = \frac{\pi \times r \times r \times 2 \times 2}{4} = \frac{\pi \times 2r \times 2r}{4}$ but 2r = d

$\therefore A = \frac{\pi \times d \times d}{4} = \frac{\pi d^2}{4}$

From $C = 2\pi r$, we can say that $\frac{C}{2} = \pi r$. We can also say that $\frac{C}{2} = \pi r \times \frac{r}{r}$.

$\therefore \frac{C}{2} = \frac{\pi r^2}{r}$ But $\pi r^2 = A$

$\Longrightarrow \frac{C}{2} = \frac{A}{r}$ or $\frac{1}{2}C = \frac{A}{r}$

From the deduction above, we can make the following implications

1) $A = \frac{Cr}{2}$
2) $C = \frac{2A}{r}$
3) $Cr = 2A$

Example 3

Find the area of a circle of radius 5cm

Solution

The area of a circle of radius r is given by $A = \pi r^2$, $r = 5cm, \pi = \frac{22}{7}$,

$\therefore A = \frac{22}{7} \times 5 \times 5$ = 78.57cm^2

Example 4

The circumference of a circle is 44cm. Find the area of the circle. Take $\pi = \frac{22}{7}$.

Solution

$A = \frac{C^2}{4\pi}$, C = 44 $\therefore A = \frac{44^2}{4 \times \frac{22}{7}} = \frac{44 \times 44}{4 \times \frac{22}{7}} = \frac{44 \times 44 \times 7}{4 \times 22} = 11 \times 2 \times 7 = 154cm^2$

Example 5

The diameter of a circle is 14cm. Find the area of the circle given that $\pi = \frac{22}{7}$.

Solution

A = $\frac{\pi d^2}{4}$, d = 14, $\pi = \frac{22}{7}$.

A = $\frac{\frac{22}{7} \times 14^2}{4} = \frac{22 \times 14 \times 14}{4 \times 7}$ = 22 x 7 = 154cm^2.

In the bid to solve the questions, find and use the result to find the required parameter. (i.e. A or C)

Length of an Arc

Draw a circle of radius 6cm and cut it out.

9. Fold the cut-out into two along a diameter.
10. What do you observe?

Since the two shapes are congruent, their arcs are also congruent. That is, the length of the arc of the semicircles is half the length of the circumference of the circle.
We note also that the angle subtended at the centre by each of the semicircle is 180^0, that is half of 360^0.

$$So\ the\ ratio\ \frac{length\ of\ arc\ of\ semicircle}{length\ of\ circumference\ of\ circle} = \frac{1}{2} = \frac{180^0}{360^0}$$

If you further fold the two into two portions such that the portions fix exactly on top of each other, the fraction of each portion or sector is one quarter of the area of the circle, remember a quadrant is formed. The angle of the quadrant is 90^0, i.e. ¼ of the angle at the centre of the circle.
Note also that the length of the arc of each quadrant is ¼ of the length of the circumference of the circle.

$$\frac{length\ of\ arc\ of\ quadrant}{length\ of\ circumference\ of\ circle} = \frac{1}{4} = \frac{90^0}{360^0}$$

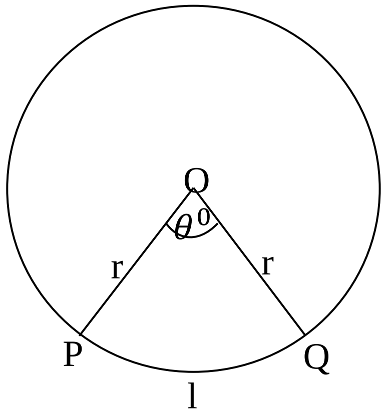


Fig. 5.38

We observe from fig. 6 that arc PQ of length l subtends an angle θ at the center of the circle, as a results

$$\frac{l}{2\pi r} = \frac{\theta}{360^0} \quad \therefore l = \frac{\theta}{360^0} \times 2\pi r$$

Example 1

An arc of length 8.5cm subtends an angle θ at the center of a circle with radius 6cm. Find the angle it subtends at the centre. Take $\pi = \frac{22}{7}$.

Solution

The angle subtended by an arc of length l with a centre of radius r is given by

$l = \frac{\theta}{360^0} \times 2\pi r, \quad 360^0 \times l = \theta \times 2\pi r$

$\Longrightarrow \theta = \frac{360 \times l}{2\pi r}$, l = 8.5cm, r = 6cm, $\pi = \frac{22}{7}$.

$\Longrightarrow \theta = \frac{360 \times 8.5}{2 \times \frac{22}{7} \times 6} = \frac{360 \times 8.5 \times 7}{2 \times 22 \times 6} = \frac{21420}{264} = 81.14^0$. The angle therefore is 81.14^0

Example 2

The radius of a circle is 21cm. An arc subtends an angle of 81^0 at the centre. Find the length of the arc. $\left(Take\ \pi = \frac{22}{7}\right)$

Solution

Let $l$cm be the length of the arc.

$\frac{l}{2\pi r} = \frac{81^0}{360^0}$

$l = \frac{81^0}{360^0} \times 2 \times \frac{22}{7} \times 21cm = \frac{297}{10}cm$ =29.7cm

The length therefore is 29.7cm

Example 3

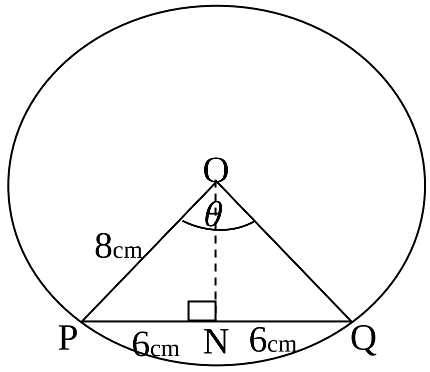


Fig.5.39

PQ is a chord of a circle, centre O. The radius of the circle is 8cm. $|PQ| = 12cm.$ Find

a) The angle subtended at the center by the minor arc PQ
b) The perimeter of the minor sector OPQ
c) The perimeter of the minor segment. (Take π = 3.142)

Solution

a) Triangle POQ is isosceles. The perpendicular $\overline{ON}$ bisects $\overline{PQ}$ $and\ \angle POQ = \theta$
$sin\frac{1}{2}\theta = \frac{6}{8} = 0.75$
$\frac{1}{2}\theta = 48.59$
$\theta = 97.18^0$ The angle therefore subtended at the centre is 97.18^0
b) If *l* is the length of the arc, then
$\frac{l}{2\pi r} = \frac{97.18^0}{360^0}$
$l = \frac{97.18 \times 16 \times 3.142}{360}\ cm$ = 13.57cm.
Perimeter of the sector = 2 x 8 + 13.57cm = 16 + 13.57cm = 29.57cm
c) Perimeter of segment = 12 + 13.57cm = 25.57cm

Area of a Sector of a Circle

We have already found out that the angle subtended by an arc at the centre of a circle is proportional to the length of the arc. Similarly, the angle subtended at the centre, i.e. the sectorial angle is proportional to the area of the sector. Considering a circle of radius r, if the area of the sector is A and the sectorial angle is θ, then

$\frac{A}{\pi r^2} = \frac{\theta^{\circ}}{360^0}$ and thus $A = \frac{\theta}{360} \times \pi r^2$

Also we know that if *l* is the length of the arc of the sector,

$\frac{l}{2\pi r} = \frac{\theta^0}{360^0}$ and $\frac{A}{\pi r^2} = \frac{\theta}{360}$. By implication we can say that

$\frac{A}{\pi r^2} = \frac{l}{2\pi r}$ and therefore $A = \frac{1}{2}lr$

Example 1

Calculate the area of a sector of a circle with radius 8cm which subtend an angle of 60^0 at the centre O.

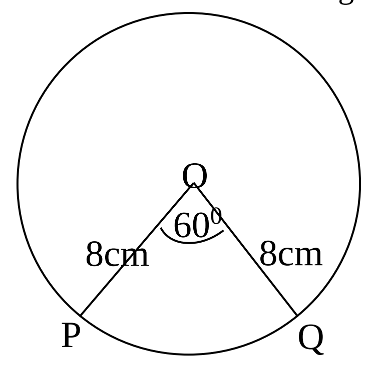


Fig. 5.40

Solution

The area of a sector of a circle is given by $A = \frac{\theta}{360} \times \pi r^2$, in our case, $\theta = 60^0$, r = 8, $\pi = \frac{22}{7}$

Area of the sector $A = \frac{60^0}{360^0} \times \frac{22}{7} \times 8^2 = \frac{84480}{2520} cm^2 = 33.52\, cm^2$

Example 4

The area of a sector of a circle with radius 10cm is $98.5cm^2$. Find the angle it subtends at the centre.

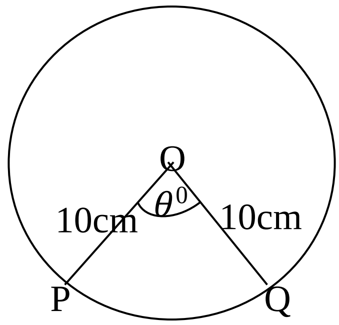


Fig. 5.41

Solution

The area of a sector of a circle with radius r is given by

$A = \frac{\theta}{360} \times \pi r^2$,

Where A = $98.5cm^2$, r = 10cm

$\therefore\ \theta \times \pi r^2 = A \times 360^0$

$\theta \times \pi r^2 = 98.5cm^2 \times 360^0$

$\theta = \frac{98.5 \times 360^0 \times 7}{22 \times 10 \times 10} = \frac{248220}{2200} = 112.83^0$

Example 3

From the diagram, calculate

i. The length of arc PQ

ii. The area of sector OPQ

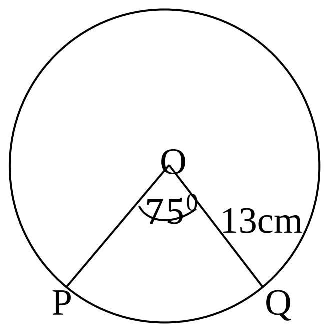


Fig. 5.42

Solution

i. The length of arc PQ is given by $l = \frac{\theta}{360^0} \times 2\pi r$

$\theta = 75^0,\ r = 13cm$

The length of arc PQ = $\frac{75^0}{360^0} \times 2 \times \frac{22}{7} \times 13 = \frac{715}{42} cm = 17.02cm$

ii. The area of a sector is given by $A = \frac{\theta}{360} \times \pi r^2$

Area of sector OPQ = $\frac{75^0}{360^0} \times \frac{22}{7} \times 13^2$ = $\frac{9295}{84}$ cm^2 = 110.65cm^2

Example 5

An arc AB subtends an angle of 100^0 at the centre of a circle with radius 14cm. Calculate the area of the major sector AOB taking $\pi = \frac{22}{7}$.

Solution

The major arc subtends an angle of (360 – 100) = 260^0 at the centre of the circle.

Area of major arc A = $\frac{260^0}{360^0} \times \frac{22}{7} \times 14^2 = \frac{4004}{9} cm^2 = 444.89 cm^2$

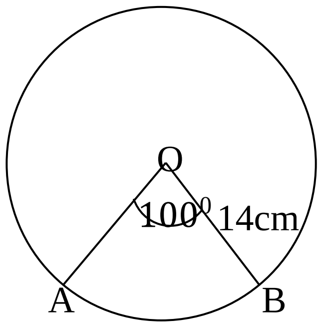


Fig. 5.43

Example 6

Find the area of the shaded portion in Fig.3. 14

Solution

The area of the square = 5 x 5 = 25cm^2

The area of the quadrant = $\frac{1}{4}\pi r^2$ = $\frac{1}{4} \times \frac{22}{7} \times 25cm^2$ = $\frac{275}{14} cm^2$ = 19.64 cm^2

So the area of the quadrant is 19.64cm^2

Area of shaded portion = area of square – area of quadrant

= 25 – 19.64 = 5.36cm^2

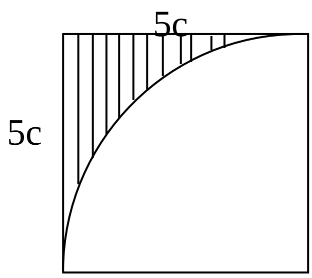


Fig. 5.44

Example 7

If a circle with a radius of 9cm has its area of sector to be 55.08cm^2, find the length of the arc which it subtends at the centre.

Solution

The relationship between the length of an arc and the area of its sector is given by $A = \frac{1}{2} lr$

Where A = 55.08cm^2, r = 9cm

$55.08cm^2 = \frac{1}{2} \times 9 \times lcm$

$l = \frac{55.08 \times 2}{9}$ = 12.24cm

CYLINDERS

r is the radius of the base of the cylinder
h is the height of the cylinder.

A cylinder has two plane circular ends each of area πr^2
The net of both ends closed, one end closed and both ends open cylinders are as shown in Fig. 4.40, Fig. 4.41 and Fig. 4.42 respectively.

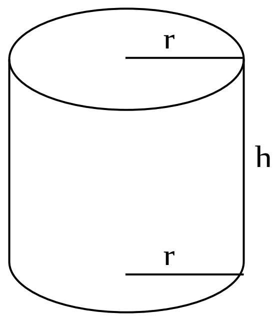
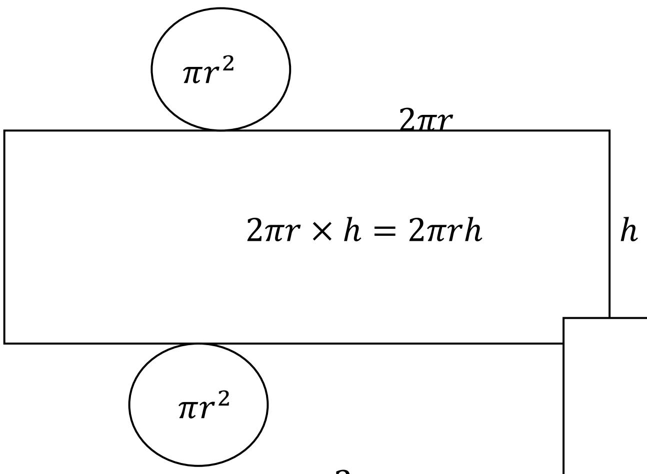


Fig. 5.44



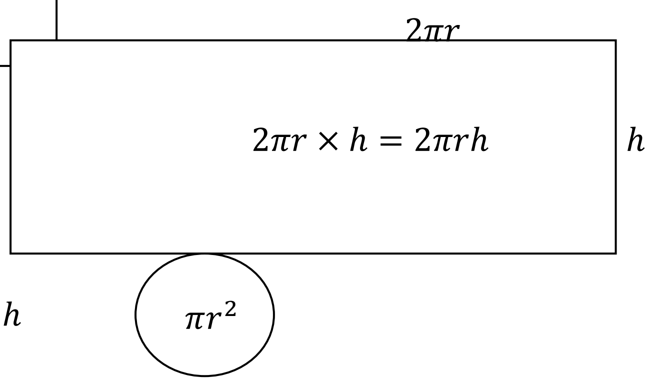


Fig. 5.44

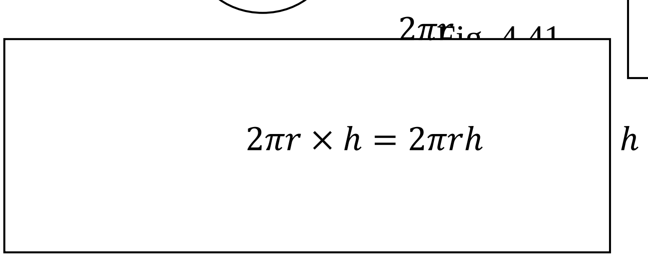


Fig. 5.43

The total surface area = curved surface area only $= 2\pi r \times h = 2\pi rh$
That is a cylinder with both ends open

For a one end open and the other closed cylinder, as shown in Fig. 4.31,
Total surface area A is given by area of base + area of curved surface. $A = \pi r^2 + 2\pi rh = \pi r(r + 2h)$

For a both end closed cylinder like we have in Fig 4.32, the total surface area A will be the sum of the three parts.
$A = 2 \times \pi r^2 + 2\pi rh = 2\pi r(r + h)$
The volume V of the cylinder = the base area x the height
But the base area $A = \pi r^2$, therefore, the volume V of a cylinder is given as
$V = \pi r^2 \times h = \pi r^2 h$

Example 15

An open cylindrical tin has a height of 60mm and a diameter of 154mm. Calculate

a. The area of the cross section
b. The volume of the cylinder
c. The area of the curved surface
d. The total surface area. $\left(take\ \pi = \frac{22}{7}\right)$

Solution

a. The radius $(r) = \frac{154}{2}mm = 77mm$

The area of the cross section $= \pi r^2 = \frac{22}{7} \times 77^2 = 18634mm^2$

b. The volume of the tin = cross sectional area x height

$= (18634 \times 60)mm^3 = 1118040mm^3$

c. The curved surface area $= 2\pi rh = 2 \times \frac{22}{7} \times 77 \times 60 = 29040mm^2$

d. Total surface area =cross sectional area + curved surface area

$$= (18634 + 29040)mm^2 = 47674mm^2$$

Example 16

The radius and height of a right circular cylinder are 7cm and 10cm respectively. Find its

(i) curved surface area (ii) total surface area, and the (iii) volume

Solution

(i) curved surface area $= 2\pi rh$

$$= 2 \times \frac{22}{7} \times 7 \times 10\ cm^2 = 440\ cm^2$$

(ii) Total surface area = $2\pi rh + 2\pi\, r^2$

$$= \left(2 \times \frac{22}{7} \times 7 \times 10 + 2 \times \frac{22}{7} \times 7 \times 7\right) cm^2$$

$$= 440\ cm^2 + 308\ cm^2 = 748\ cm^2$$

(iii) $volume = \pi r^2 h$

$$\left(\frac{22}{7} \times 7 \times 7 \times 10\right) cm^3 = 1540cm^3$$

$$= 1540\ cm^3$$

Example 17

Radius of a road roller is 35 cm and it is 1 metre long. If it takes 200 revolutions to level a playground, find the cost of leveling the ground at the rate of £3 per m^2.

Solution:

Area of the playground leveled by the road roller in one revolution

= curved surface area of the roller

$= 2\pi rh = \left(2 \times \frac{22}{7} \times 35 \times 100\right) cm^2 \quad (r = 35\ cm,\ h = 1,\ m = 100\ cm)$

$= 22000\ cm^2$

$= \left(\frac{22000}{100 \times 100}\right) m^2$

(since 100 cm = 1 m, so 100 cm × 100 cm = 1 m × 1 m)

$$= 2.2\ m^2$$

Therefore, area of the playground levelled in 200 revolutions $= 2.2 \times 200\ m^2 = 440\ m^2$

Hence, cost of leveling at the rate of £3 per m^2 = £3 × 440 = £1320.00

Try these

1. The length and breadth of a cuboidal tank are 5m and 4m respectively. If it is full of water and contains 60 m3 of water, find the depth of the water in the tank.
2. A hollow cylindrical metallic pipe is open at both the ends and its external diameter is 12 cm. If the length of the pipe is 70 cm and the thickness of the metal used is 1 cm, find the volume of the metal used for making the pipe.
3. A metallic solid of volume 1 m3 is melted and drawn into the form of a wire of diameter 3.5 mm. Find the length of the wire so drawn.
4. Find the curved surface area, total surface area and volume of a right circular cylinder of radius 5 m and height 1.4 m.
5. Volume of a right circular cylinder is 3080 cm3 and radius of its base is 7 cm. Find the curved surface area of the cylinder.
6. A cylindrical water tank is of base diameter 7 m and height 2.1 m. Find the capacity of the tank in litres.
7. Length and breadth of a paper is 33 cm and 16 cm respectively. It is folded about its breadth to form a cylinder. Find the volume of the cylinder.
8. A cylindrical bucket of base diameter 28 cm and height 12 cm is full of water. This water is poured in to a rectangular tub of length 66 cm and breadth 28 cm. Find the height to which water will rise in the tub.
9. A hollow metallic cylinder is open at both the ends and is of length 8 cm. If the thickness of the metal is 2 cm and external diameter of the cylinder is 10 cm, find the whole curved surface area of the cylinder (use π = 3.14).

[**Hint:** whole curved surface = Internal curved surface + External curved surface]

Pipes

A pipe is a hollow which is cylindrical in shape and solid as in Fig. 5.81

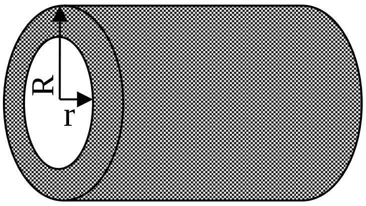


Fig. 5.45

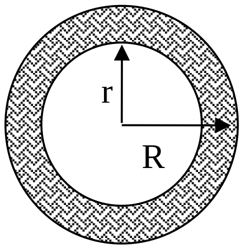


Fig. 5.46

The cross section of the pipe is annular as shown in Fig. 4.44. if an external radius of the cross section is R and the internal radius of the cross section is r, then the area of the cross section is

$A = \pi R^2 - \pi r^2 = \pi(R^2 - r^2)$

If h is the height or length of the pipe, then the volume V is given as

$V = \pi(R^2 - r^2)h = \pi h(R^2 - r^2)$

Example 18

A copper pipe has an external radius of 100mm and internal radius of 80mm.

Find the cross sectional area of the pipe $(take\ \pi = 3.142)$
Calculate the volume of copper needed to make 5m length of this pipe

Solution

External radius (R) = 100mm, internal radius (r) = 80mm.

The area of the cross section of the pipe = $\pi(R^2 - r^2) = 3.142(100^2 - 80^2)$

$= 3.142 \times 3600 = 11311.2mm^2$

The volume of copper needed to make 5000mm length is
Area of cross section x length
$= 11311.2 \times 5000 = 56556000mm^3$

Try these

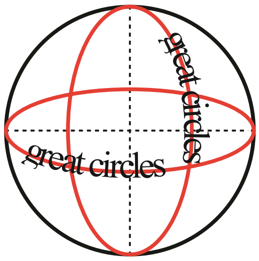
1. A rectangular sheet of metal made of uniform material is 9cm long and 8cm wide. Six circular holes of radius *r* cm are drilled through the sheet.
2. Find an expression for the volume of the metal left after the drilling, if *h* cm is the thickness of the metal.
3. If the ratio of the new weight of the sheet to the original weight is 13:20, find, correct to two decimal places, the value of *r*. (take π = 3.14

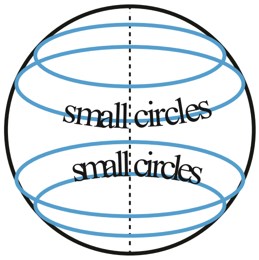
STRAND 6 GLOBAL MATHEMATICS:

LEARNING, TEACHING AND APPLYING

The Earth as a sphere

In this topic, the Earth will be considered to be a perfect sphere, with a radius of 6400 kilometres. There are two types of circles that can be drawn on the surface of a sphere.





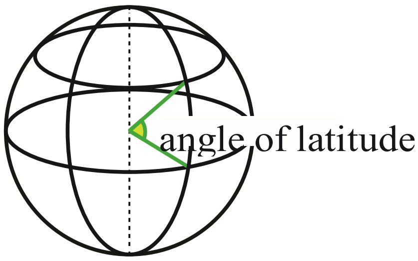
A **great circle** is the largest circle that can be drawn on the surface of a sphere, as the centre of the great circle is also the centre of the sphere. The radius of a great circle on Earth will therefore be 6400 kilometres which is the same as the radius of the Earth sphere. As shown from the figures, the equator and a circle drawn through both the North and South poles are great circles. The meridians of longitude are all great circles.

A great circle track is the shortest distance between two points on the surface of a sphere and forms the basis of many navigational activities for long distance travel by air or sea. A key calculation as part of this process is the finding of waypoints; these are points that the traveler should pass through to maintain their journey on the great circle.

A **small circle** does not have its centre at the centre of the sphere. Obviously, the radius of a small circle will be LESS than the radius of the sphere. For Earth, it means that the radius of a small circle will be less than 6400 kilometres. The parallels of latitude (apart from the Equator) are all small circles.

Latitude;

Latitude is a measure of the position of a point on the earth's surface in terms of degrees north or south of a baseline - the equator. These values vary from 90^o S to 90^o N. The latitude of the Equator is 0^o .



Every plane perpendicular (at right angles) to the earth's axis cuts the surface of the earth in a circle called a parallel of latitude.

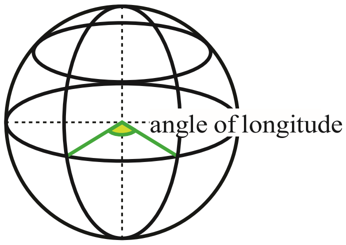
The parallel of latitude formed by a plane passing through the centre of the earth and equidistant from both the North and South Poles is called the **equator**. All other parallels of latitude are small circles, each having a radius smaller than that of the earth, i.e. less than 6400 km. As the parallels run east and west, places having the same latitude will be due east or due west of each other.

Distance north or south of the equator is calculated from the arc length using the angle between two radii - one going to the equator and the other going to the point whose latitude is required.

Latitude is determined by measuring the angle of elevation of the sun at noon, and adding or subtracting an allowance for the time of year. Using the angle only is correct for just two days of the year - the spring and autumnal equinoxes. In midwinter we need to subtract 23.45^o and at midsummer we need to add 23.45^o These allowances are due to the fact that the earth is tilted at this angle as it orbits the sun. Books of correction factors for every day of the year (almanacs) were used for many years until global positioning satellites were used.

Longitude;

In the same way that latitude fixes a place north or south of the equator, **longitude** fixes a place east or west of the prime meridian - also known as the Greenwich meridian which passes through Greenwich near London, UK.



Each great circle passing through the north and south poles has the earth's axis as its diameter. Each great circle is divided by the poles into two semi-circles called **meridians of longitude.** As the meridians run north and south, places having the same longitude will be due N or due south of each other.

Distance east or west of Greenwich is measured as an angle between the plane containing the Greenwich meridian and the plane containing the meridian passing through the point whose longitude is required.

Longitude varies from 180^o W of Greenwich to 180^o E of Greenwich.

The two meridians on a particular great circle will have values which sum to 180^o, but one will have a designation W and the other a designation E. For example, 60^o W is on the same great circle as 120^o E.

The accurate determination of longitude was one of the great technological challenges of the eighteenth century, since a number of shipping disasters had shown up the inadequacies of the methods being used at the time. A number of European nations offered financial prizes for the development of a reliable and accurate method for determining longitude.

The Earth rotates at a rate of 360^o per day, or 15^o per hour, so there is a direct relationship between time and longitude. If a navigator knew the time at a particular fixed reference point when the local time could be determined at the ship's location, the difference between the reference time and the apparent local time would give the ship's position relative to the fixed location.

Measuring the Lunar Distance

The angle between the Moon and another celestial object (a star or the sun) - which would lead to determination of the position of the measurer through calculation based on reference values is termed as measuring the lunar distance. This method required the compilation of tables of reference values (an almanac) for every day of every year well in advance of their required use - a task which required a lot of human calculating time in a pre-electronic computer era.

The use of Chronometer

The second, and simpler, method involved taking a chronometer, which could keep very accurate time at sea, and calculate position almost immediately by comparing the chronometer time for the fixed reference point with local time. It took until the 1850's to build accurate chronometers cheap enough for their widespread use.

The advent of radio in the early twentieth century enabled navigators to verify the accuracy of their chronometers with time signals broadcast from known locations. Global positioning satellite systems and radar have added to the tools available to the modern navigator.

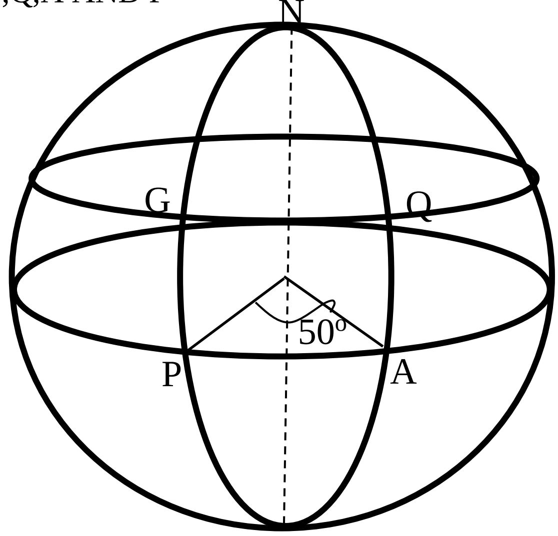
Locating positions on Earth

The position of any point on the earth's surface is given uniquely by the intersection of the circles of latitude and longitude.

By convention, the latitude is given first when giving the position of any point. In the diagram, the meridian through NGPS is the Greenwich meridian.

Example

Points G and Q are on the 40^o N parallel of latitude. Points P and A are on the equator. Write down the positions of points G,Q,A AND P



Solution

In the diagram, the meridian through NGPS is the Greenwich meridian.

G (40^o N, 0^o E (or W))

Q(40^o N, 50^o E)

A (0^o N, (or S) 50^o E)

P(0^o N (or S), 0^o E)

Distance along meridians

Example

Paris is at 48.67^o N and 2.33^o E. How far is it from Paris to the North Pole and the equator travelling along the meridian? Correct to the nearest km.

Solution

Each meridian is a great circle, with a radius of 6400km. The angle between the latitude of Paris and that of the Equator is 48.67^o. the angle between the latitude of Paris and the North Pole 90^o – 48.67^o.

From Paris to the equator

$l = \frac{\emptyset}{360} x\ 2\pi r = \frac{\pi r}{180}$ x $\emptyset$

= $\frac{6400\pi}{180} x\ 48.67^o$

5436.491…..

= <u>5436km</u>

From Paris to the North Pole

$l = \frac{\emptyset}{360} x\ 2\pi r = \frac{\pi r}{180}$ x $\emptyset$

= $\frac{6400\pi}{180} x\ 41.33^o$

4616.605…..

= <u>4617km</u>

Note: The metre was originally defined as one ten-millionth of the distance from the North Pole to the Equator travelling along the meridian through Paris. Owing to some errors in estimating the shape of the earth, the defined metre was about one-fifth of a millimetre shorter than the actual distance, meaning that the actual circumference of the earth through the poles is 40 007 863 m rather than the expected 40 000 000 m.)

Example

Melbourne is at 37.82 S and 144.97 E. How far is it from Melbourne to the South Pole, the equator and the North Pole travelling along the meridian correct to the nearest kilometer.

Solution

Each meridian is a great circle, with a radius of 6400km.

From Melbourne to the equator.

The angle between the latitude of Melbourne and that of the Equator is 37.82°.

$l = \frac{\emptyset}{360} \; x \; 2\pi r = \frac{\pi r}{180} \text{ x } \emptyset$

$= \frac{6400\pi}{180} \; x \; 37.82^{o}$

4224.53…..

= <u>4225km</u>

From Melbourne to the South Pole, the angle between the latitude of Melbourne and the South Pole is $90^{o} - 37.82^{o} = 52.18^{o}$

$l = \frac{\emptyset}{360} \; x \; 2\pi r = \frac{\pi r}{180} \text{ x } \emptyset$

$= \frac{6400\pi}{180} \; x \; 52.18^{o}$

5828.56…..

= <u>5829km</u>

From Melbourne to the North Pole, the angle between the latitude of Melbourne and the North Pole is $90^{o} + 37.82^{o} = 127.82^{o}$

$l = \frac{\emptyset}{360} \; x \; 2\pi r = \frac{\pi r}{180} \text{ x } \emptyset$

$= \frac{6400\pi}{180} \; x \; 127.82^{o}$

14277.63…..

= <u>14278km</u>

Distances between two places

Finding the distance between two places on the surface of the Earth implies finding either the great circle distance (the shortest distance between any two points on the surface of a sphere) or the small circle distance (travelling along a parallel of latitude). To complete these calculations, we need to know the location (latitude and longitude) of each place. For this course, we will find the following;

- The first is finding the great circle distance between two places on the same meridian of

longitude, or two places on the Equator.

- The second is finding the small circle distance between two places on the same parallel of latitude.

Distances between places with the same longitude
Example

Both Torrens Creek and Kyabram are on the 145^o E meridian of longitude, but Torrens Greek is at 20 .77 S whereas Kyabram is at 36.32^o S. How far is Torres Creek from Kyabram travelling along the 145^o E meridian correct to the nearest km.

Solution

Each meridian is a great circle, with a radius of 6400km. The angle between the latitude of Torres Creek and that of Kyabam is 36.32 – 20.77^o = 15.55. (We find the difference since both places are on the same side of the Equator)

From Torres Creek to Kyabam

$$l = \frac{\emptyset}{360} x\ 2\pi r = \frac{\pi r}{180} \text{ x } \emptyset$$

$$= \frac{6400\pi}{180} x\ 15.55^o$$

1736.95…..

= <u>1737km</u>

Example

Both Cooktown (Queensland) and Kyabram (Victoria) are on the 145 degrees E meridian of longitude, but torrens creek is at 20.77 degrees S where Kyabram is at 36.32 degrees S. how far is it from torrens creek to kyabram travelling across the 145 degrees E meridian correct to the nearest km?

Solution

Each meridian is a great circle with a radius of 6400 km. the angle between the latitude of torrens creek of the kyabram is 36.32 – 15.47 = 20.85

(we find the difference since both places are on the same side of the equator)

From torrens creek to kyabram

l = $\frac{\emptyset}{360}\ x\ 2\pi r = \frac{\pi r}{180}$ x $\emptyset$

= $\frac{6400\pi}{180}\ x\ 20.85$

2328.967…..

= <u>2329 km</u>

Example

Both shellharbour (NSW) and Magadan (Russia) are on the 151 latitude but shellharbor is at 34.58 degrees S whereas Magadan is at 34.58 degrees S whereas Magadan is 59.57 degrees N. how far is it from Shellharbour to Magadan travelling along the 151degree E meridian, correct to the nearest km.

Solution

Each meridian is a great circle with a radius of 6400 km. the angle between the latitude of shellarbour and that of magadan is 34.58 + 59.57 = 94.15

(We find the sum since both places are on different sides of the equator)

From shellarbour to magadan

l = $\frac{\emptyset}{360}\ x\ 2\pi r = \frac{\pi r}{180}$ x $\emptyset$

= $\frac{6400\pi}{180}\ x\ 94.15$

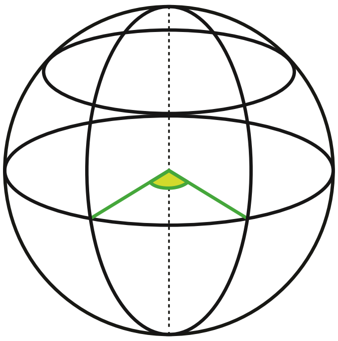
10516.655…..

= <u>10517 km</u>

Distance between places on the Equator

The Equator is a great circle, and hence finding the distance between two points on the Equator uses the same calculation process as for finding the distance between two points on the same meridian of longitude.

difference in angle of longitude



Both Libreville and Kismanyo are on the Equator on opposite of the Africa continent. Libreville is at $9.27^{o\ E}$ and Kismanyo is at 42.32^o E. How far from Libreville to Kismanyo travelling along the equator, correct to the nearest km.

Solution

The equator is a great circle with radius of 6400km. the angle between the longitude of Libreville and that of Kismanyo is$42.32^o - 9.27^o = 33.05$

(we find the difference since both places have the same longitude direction, (**E**)

From Libreville to Kismanyo

$$l = \frac{\emptyset}{360}\ x\ 2\pi r = \frac{\pi r}{180} \text{ x } \emptyset$$

$$= \frac{6400\pi}{180}\ x\ 33.05$$

3691.720…..

= <u>3692 km</u>

Example

Both the Galapagos Islands and the island of Naura are on the Equator, but the Galapagos Islands are at 90.30^o W whereas the Island of Nauru is at 166.56^o E. How far is it from Galapagos Island to Nauru travelling over the Pacific Ocean along the equator, correct to the nearest km?

Solution

The angle between the Longitude of the Galapogas Island and that of Nauru is $90.30 + 166.56^o = 256.86^o$

(we find the sum since these places have different longitude directions.

But this is a major arc and the minor arc will be $360 - 256.86 = 103.14$

We could also find the angle between these two places by recognising that both are close to 180^o E/W. we could find the angle between the Galapagos island and 180^oE/W, the island of Nauru and 180^o E/W and then add these two angles together.

Angle between Galapogos island and 180^o E/W = $180 - 90.30 = 89.70$

Angle between Nauru island and 180^o E/W = $180 - 166.56 = 13.44$

Total angle between Galapagos island and Nauru = $89.70 + 13.44 = 103\ 14$

From the Galapagos island to Nauru

$$l = \frac{\emptyset}{360}\ x\ 2\pi r = \frac{\pi r}{180} \text{ x } \emptyset$$

$$= \frac{6400\pi}{180}\ x\ 103.14$$

11520.848…..

= <u>11521 km</u>

STRAND 7 INTRODUCTORY STATISTICS:

LEARNING, TEACHING AND APPLYING

DATA

A collection of measurements or attributes items or individuals is referred to as data. It could be seen that, when we talk of data we are talking about a collection; plural and not singular. If you therefore collect an information on an item, it is not a data but a **variable**. Every individual possesses a number of characteristics or attributes. Examination marks, age, colour, height, weight and so on are all characteristics or attributes. Characteristics can assume different values, categories or descriptions.

Types of Variables

There are broadly two main types of variables; Quantitative variable and Qualitative variable.

Quantitative Variable

Any activity that involves counting or measuring is said to be quantitative. A quantitative variable is an observation that can therefore be counted or measured. They are said to be numerical. Quantitative data therefore are observations that are measured or counted. If an observation is made, we say it is a quantitative variable. Example, the mass of a girl in a class is a quantitative variable, but the collection of masse of girls in a class is a quantitative data. Quantitative variables are either ***discrete*** or ***continuous.***

Values of **discrete data** are usually found by counting. They are a finite or countable number of choices. The data are restricted to only certain, or exact numbers. For example, number of pupils, number of cars, number of animals, which can be 0, 1, 2, 3, …. They are not measurement. We can't have negative number of pupils or animals therefore do not involve negatives.

Values of **continuous data** can be any real number. They are normally found by measuring; length, mass, distance, time etc. With continuous data, all decimals or fractions are taking into consideration. It is also rounded to nearest degree of accuracy.

Qualitative variable

This is an operational concept. Qualitative variables cannot be measured or counted without introducing coding. They are measurements which fall into various categories which are exclusive. This can also be classified as categorical variables or attribute variables. They therefore consist of counts or frequencies in the various categories. Some examples are gender, religion, political party, colours, etc.

MEASUREMENT SCALES

We apply numbers in a range of different ways that show they have several symbolic uses. **Data** can be classified according to the following levels of measurement which also represent the various numbers we can think of. The levels of classifications are **Nominal, Ordinal, Interval and ratio**.

Nominal numbers

This is the most limited level of measurement. It applies to qualitative data only. Items can be described or identified using unique numbers. This is just like your house number; in that it uniquely identifies your property from those of your neighbours. We can as well use other form of unique identifier, such as a letter or even words. Numbers have a very useful property: you cannot run out of them; do you see the difference between your index number and your friends'?. Letters and combinations of letters on the other hand, becomes cumbersome when you need to uniquely identify large quantities of individual items. You cannot perform any arithmetical or mathematical operations on them. When numbers are used in this way, they are referred to as **nominal.**

On the nominal scale, no order is required. The categories are mutually exclusive and exhaustive. Categories are said to be mutually exclusive if an individual or item must belong to one and only one category. In the same way, categories are said to be exhaustive if an individual must belong to at least one of the categories. Examples of nominal measurements are gender; we can list categories as male and female or female and male, marital status, type of cars, buildings, schools, etc (they don't have any particular order).

Ordinal

The next higher scale of measurement is the ordinal scale. On the ordinal scale, order is necessary, meaning one category is lower than the next one or vice versa. For instance, we have the order of occurrence like 1^{st}, 2^{nd}, 3^{rd} … This is basically based on position. Examples are the GES ranking thus teacher, supt.2, supt.1, principal supt., etc, in the Ghana Army the rank of lieutenant is lower than the rank of a captain, which is in turn lower than the rank of major and so on. Grades are also ordinal; as excellent is higher than very good, which in turn is higher than good, and so on.

Despite of the positions as 1^{st}, 2^{nd}, 3^{rd}, and so on, the differences cannot be found or determined. This is because, if we rank someone as 1^{st}, it does not mean he is twice or thrice better than the one who is 2^{nd} or 3^{rd}.

Interval

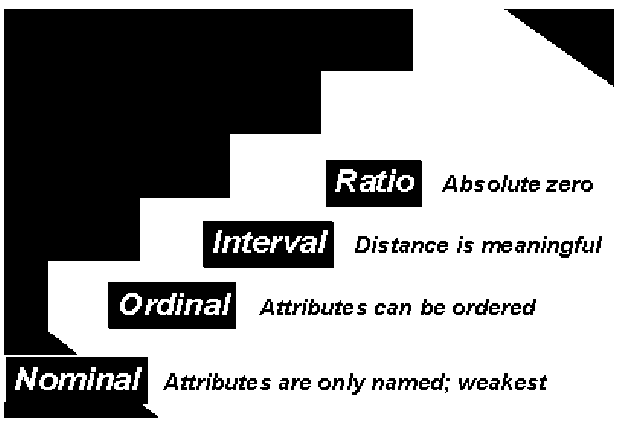
The interval scale is the next higher scale of measurement. These are numbers that actually represent quantities therefore applies to quantitative data only. It has all the properties of the ordinal scale, with the additional property of finding a meaningful amount of differences between values. Here, distance between values is constant. There is no natural zero starting point. It is arbitrary. Examples of variables on an interval scale is temperature. On the temperature for instance, a country ‘A’ can be 30 degrees cold and another country ‘B’ be 15 degrees cold but that does not mean country ‘B’ is as twice colder than country ‘A’. This is because, on the interval scale, the ratio between two numbers is not meaningful.

Ratio

The ratio scale is the highest scale among all the measurement scales. It applies to quantitative data only and has all the properties of the interval scale. In addition to these properties, the ratio scale has a meaningful zero starting point and a meaningful ratio between two numbers.

An example of variables measured on the ratio scale is mass. A weighing scale that reads 0kg gives an indication that there is absolutely no mass on it, making the zero starting point meaningful. A bag ‘A’ of rice weighs 50kg and another bag ‘B’ of rice weigh 10kg, then it is reasonable to say that Bag ‘A’ weighs five times than bag ‘B’. Hence the ratio between two numbers is meaningful.

LEVELS OF MEASUREMENT



SOURCES OF DATA

Sources of data can be put into two main categories, depending on their originality. These are primary sources and secondary sources. Data from a primary source are called primary data and those from a secondary source are called secondary data.

I. Primary Sources of Data

Primary data are collected by the researcher/investigator/surveyor. There are three basic methods of obtaining primary data: **observation, surveys and experiment.** The choice of method is influenced by the nature of the problem and by the availability of money and time.

A. Observation

In the observation method, a situation of interest is checked by a person or some mechanical which records the relevant facts, action, or behaviours. For example through observation we are able to obtain accurate data about what consumers do in certain situations. However observation does not tell why an event happened. There are also two types of observation; participatory observation and nun- participatory observation. In the participatory, the researcher takes part in the activity whilst in the non-participatory, and the researcher does not take part of the activity.

B. Survey

In surveys the researcher's task is to find a way of obtaining information from individual often referred to as respondents. A survey conducted on an entire population of interest is called a census and a survey conducted on a sample is called a sample survey. Most of the times, a questionnaire is used to obtain the information from respondent. The questionnaire may be administered by post, by telephone via internet or in person.

C. Experiment

With this, data are generated by the researcher through an experiment. Experimental research is concerned cause-and- effect relationship. An experiment can be conducted either in a laboratory or in a field setting. In a laboratory experiment, the researcher has complete control during the experiment. A field experiment is conducted under more realistic conditions.

II. Secondary source of data.

Secondary data are those that have already been gathered or published for some other purpose. This is faster to collect and less expensive than primary data. Sources of secondary data include those inside the organization (internal) and those outside the organization (external). Secondary data are available from libraries, government agencies and the internet.

A. Libraries

A common place to look for secondary data is a library. Periodicals (news papers, magazines, journals, etc) are materials that are published at regular intervals (daily, monthly ,quarterly ,etc)

B. Government agencies

Government data are available to in publications issued by local, state national or international government. Government data are generally considered as being reliable and include laws, regulations, statistics, consumer information.

C. Internet

We can search for secondary data using a variety of resource discovery tools Called search engines, such as yahoo, Google, etc on the internet.

REPRESENTATION OF DATA

Introduction

Data can be represented in different ways. It can be represented on a frequency distribution table, graphically, pictorially etc. In a distribution, it is important to know the frequency of the distribution. The number of occurrence of individual items that fall under a particular class is its **frequency**.

Frequency distribution table.

This is the tabular arrangement of data by grouping them into classes together with corresponding class frequencies. When repeated observations are made on a variable (which can be marks, heights, ages etc.) the results and analysis is a **frequency distribution**. A frequency distribution is recorded in a frequency table. The values of the variable are sometimes grouped into categories showing the number of observations belonging to each mutually exclusive category called classes, depending on the data. The table gives the possible value or class intervals of the variables and the corresponding frequencies.

Representation of data using frequency distribution table

As explained earlier, frequency distribution table is a tabular arrangement of data by grouping them into classes together with corresponding class frequencies.

The tally table is a simple frequency table which contains the tally marks. The tally is normally done on a large distribution so that the investigator does not lose count of the variables.

A frequency table may not necessarily contain the tally mark nor the name frequency. For instance, the table below does not contain the name frequency. The frequency is represented by the number of boys.

Fillings	0	1	3	3	4
Number of boys	9	15	13	5	3

Ungrouped and Grouped frequency distribution

Ungrouped frequency distribution.

This is a distribution organized in such a way that the individual items or marks are not grouped in any way, they stand on their own for analysis. This mostly happens when the data is qualitative though it can be quantitative. Each class in the table consists of a single value. For instance when given;

Scores obtained by 1B class

Marks	Frequency
4	5
5	2
6	3

This is an ungrouped data where the items 4, 5 and 6 are standing on their own for the distribution analysis.

Ungrouped data is mostly used when the items in the distribution are few.

Example:

Supposing a raw data is given below

13 9 15 17 13
9 11 9 11 13
17 15 11 9 9
11 15 11 11 11

A simple frequency distribution table which contains the tally marks can be created as below.

Ungrouped frequency distribution table

Ages of 1B class

Ages (x)	Tally	Frequency
9	~~////~~	5
11	~~////~~ //	7
13	//	2
15	////	3
17	//	2

Grouped frequency distribution

Grouped frequency distribution becomes very important when there are many frequencies in a distribution and, or involves a large range of values.

Construction of grouped frequency distribution table involves the determination of the number of classes, the class limits, the class boundaries, the class frequencies, and so on.

Determination of Number of classes

Choosing the number of class's for a frequency distribution is fairly subjective. It should be noted, however, that too few classes tend to conceal important information about the data; and too many classes can erode the advantage it has, over a raw data. In practice, most classes are hardly fewer than five and more than ten.

Class Limits.

Each class has two limits- the lower-class limit and the upper-class limit. The lower-class limit is the smallest value that belongs to the class. For instance, when the range is between say 0-9 we say 0 is the lower class limit and the 9 is the upper class limit of the particular class. Class limits are chosen in such a way that the resulting classes are mutually exclusive (i.e. none overlapping) and exhaustive (i.e. every observation must belong to a class).

The range of possible values for a grouped or a class is its **class interval**. It is the difference between two class limits

Class frequencies.

The last step in the construction of a frequency table is to determine the class frequencies. This is a process of assigning an individual value to the data in the class that it belongs. If the data is large, it is advisable to use a tally mark (/) to represent an individual observation in the class. After every individual has been assigned to its class, the tallies for each class are totalled. These totals then become the frequencies for the classes.

Example: The number of runs scored by 36 batmen in a cricket competition is as follows:

31	40	20	35	21	12
11	39	16	0	28	49
29	32	17	2	12	24
14	1	10	23	19	13
26	30	10	21	18	0
15	11	29	8	12	1

Tally these values in a grouped frequency table using class intervals of 0-9, 10-19 etc.

Solution

No of runs	Tally	Frequency
0-9	//// /	6
10-19	//// //// ////	14
20-29	//// ////	9
30-39	/////	5
40-49	//	2

For grouped data, let us study the following:

Class Boundaries;

This separate one class from the other. For a grouped frequency, each class boundary is usually half – way between the upper class limit of one class and the lower class limit of the next class. Class boundaries or true class limits are boundaries between successful classes. Each class, therefore, has two boundaries-the lower class boundary and the upper class boundary.

A lower class boundary is obtained by adding the lower limit of the class to upper limit of the proceeding class and dividing the sum by two.

i.e. ½ (upper class limit of one class + lower class limit of the next class) for instance from the table above

For example,

$$\frac{1}{2}(9+10) = \frac{1}{2} \times 19 = 9.5$$

Giving the boundaries 0.5 – 9.5, 9.5 – 19.5 etc.

Also you can find the difference between upper class limit and its next lower class limit and divide it by two (2) e.g. 39 and 40. 40- 39. = 1 ÷ 2 = ½

After which subtract the answer from the lower class limit and add to the upper class limit of the same class. i.e. 0.5 10- 0.5 = 9.5 and 19+ 0.5 = 19.5

Study this

Marks	Frequency	Class boundaries
35-39	3	34.5-39.5
40-44	2	39.5-44.5
45-49	4	44.5-49.5

Marks	Frequency	Class boundaries
980-1060	10	940-1040
1080-1100	15	1040-1140
1180-1200	24	1140-1240

This is a clear indication that, for class boundaries, you don't always have to add or subtract 0.5 just like that, to the class limits, but rather, use the proper procedure as shown above for your correct answer.

Class size or width

This is the difference between the lower and the upper class boundaries of a class interval. For classes of equal widths, the common width is the difference between any two successive lower class limits; or any two successive upper class limit.

Class size = upper class boundary – lower class boundary. Eg from the table 35-39 is (39.5 – 34.5) = 5

It can also be the difference between successive mid values.

Class midpoint or mid-value or class marks

For grouped frequency distribution, the class mid-point or class mid value is made to stand for the class marks. This is the middle value of a class. It is obtained by adding the lower and the upper class limits and dividing the results by 2.

Also, it is the average of the lower and the upper class boundaries.

= ½ (lower class limit + upper class limit)
or ½(lower class boundary+ upper class boundary)
e.g. from the table 35-39 implies mid-point equals ½ (35+39) = 37.
This class midpoint is often used in calculating the mean and standard deviation of a grouped frequency distribution.

Relative frequency table

Class frequencies can be converted to relative class frequencies in other to show the proportion of the total number of observations in each class. To convert a frequency distribution to a relative frequency distribution, each of the class frequencies is divided by the sum of the frequencies.

For example, considering the table below,

Class limit	Class boundary	Class mark	frequency	Relative frequency
10-19	9.5-19.5	14.5	1	0.025
20-29	19.5-29.5	24.5	1	0.025
30-39	29.5-39.5	34.5	3	0.075
40-40	39.5-49.5	44.5	4	0.100
50-59	49.5-59.5	54.5	14	0.350
60-69	59.5-69.5	64.5	12	0.300
70-79	69.5-79.5	74.5	3	0.075
80-89	79.5-89.5	84.5	2	0.050
			40	**1.000**

Using the distribution table above, the relative frequency of the 60-69 class is 12/40 = 0.3

Since class frequency is 12 and the sum of frequencies is 40. Note that, relative frequencies must add up to 1, allowing for rounding errors.

GRAPHICAL REPRESENTATION OF DATA

Graphics or pictures can be used to represent frequency distribution for easy reading and analysis. Below are some of the ways we can represent information graphically.

1. Bar chart

2. Histogram

3. Pie chart

4. Cumulative frequency curve

5. Pictogram,

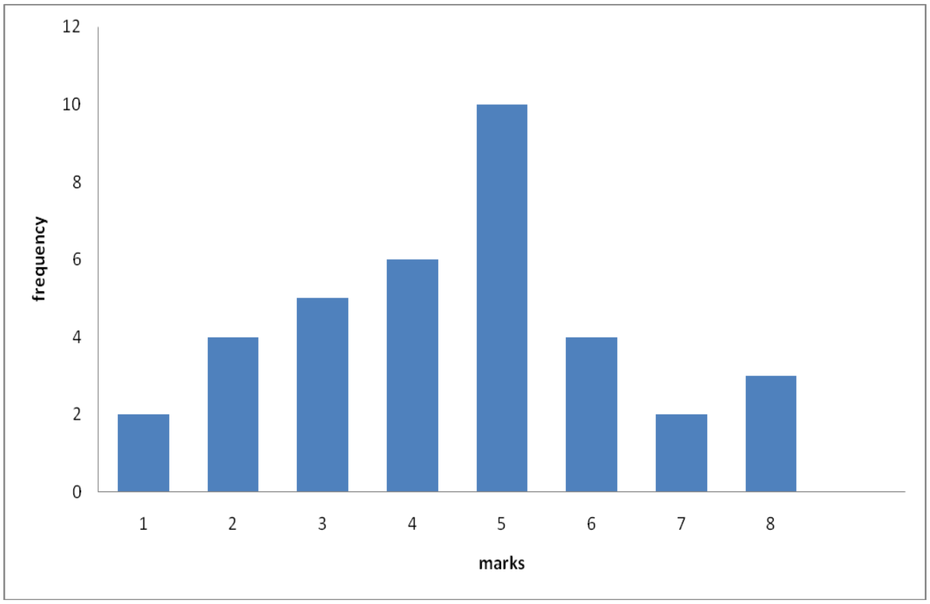
6. Frequency polygon (line graph) etc

BAR CHART

For this type of chart, the frequencies are represented by a series of parallel bars of equal width, on e bar for each category, the heights represent the frequency. Also, the bars are normally in the vertical position with equal spaces between them.

Illustration
Below is a frequency distribution table, draw a bar graph for the it.

Marks	1	2	3	4	5	6	7	8
Frequency	2	4	5	6	10	4	2	3



The above diagram represents a simple bar chart which consist of vertical bars or rectangles placed along the category axes . the height of a bar represents frequency or value corresponding to a category.

HISTOGRAM

The histogram consists of number of rectangles or bars. The centre of the rectangle or the bar is at the class midpoint value whilst the length is the class size. It is also normally in vertical position with no space between them. When presented, the difference between the bar chart from the histogram is that, the bar chart has spaces between them whilst the histogram is compact.

The histogram illustrates a frequency distribution with rectangles drawn on a continuous base. The area of each rectangle is proportional to the frequency of the class it represents. The rectangles do not have to be of equal width but their bases must be proportional to the class width. That is the extremes of the base of each rectangle are at the lower class boundary and the upper class boundary of the class it represents.

The upper class boundary of one class must coincide with the lower class boundary of the next class to ensure continuity.

Illustration

Grouped data is normally used for the presentation. Example,

Having,

Marks	No of students	Class boundary	Class midpoint
1-5	2	0.5-5.5	3
6-10	5	5.5-10.5	8
11-15	9	10.5-15.5	13

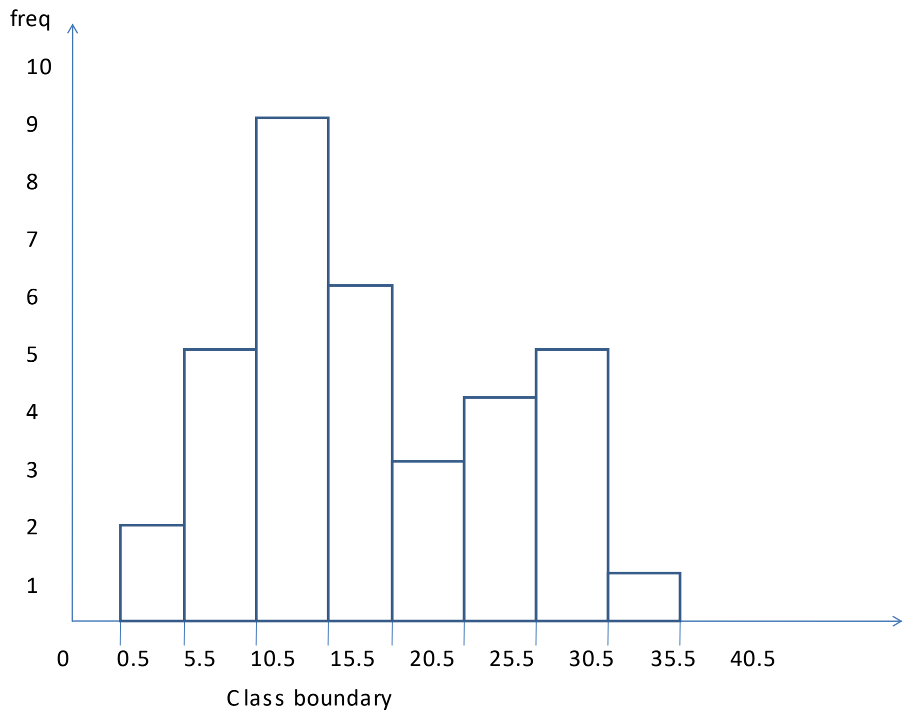
Then using the class boundaries to draw the histogram, we have 0.5-5.5, 5.5-10.5,

10.5-15.5, etc.

To draw a histogram, the first thing you do is to check the class intervals. If the class intervals are the same, then you don't have much problem so you go ahead and draw your graph taking into consideration the marks and their frequencies.

Draw a histogram to represent the following data using class boundaries.

No of students	Frequency	Class midpoint	Class boundaries
1 – 5	2	3	0.5 – 5.5
6 – 10	5	8	5.5 – 10.5
11 – 15	9	13	10.5 – 15.5
16 – 20	6	18	15.5 – 20.5
21 – 25	3	23	20.5 – 25.5
26 – 30	4	28	25.5 – 30.5
31 – 35	5	33	30.5 – 35.5
36 – 40	1	38	35.5 – 40.5



Since the area of a rectangle or a bar a histogram occupies corresponds to the class size of that particular class, it becomes very necessary to check your class intervals before starting to plot your histogram as said before. Supposing the intervals are not the same, then, some bars may be smaller, whilst some bigger, depending on the size of the class.

Study the following

No of students	Frequency	Class midpoint	Class boundaries
1 – 5	2	3	0.5 – 5.5
6 – 10	5	8	5.5 – 10.5
11 – 20	9	15.5	10.5 – 20.5
21 – 30	6	25.5	20.5–30.5
31 – 40	3	35.5	30.5– 40.5
41 – 45	4	43	40.5– 45.5
46 – 50	5	48	45.5 – 50.5
51 – 60	1	55.5	50.5 – 60.5

We could see that the class intervals are no more equal, meaning the class sizes are also not the same. In this case, to plot a histogram, you need to find the frequency density.

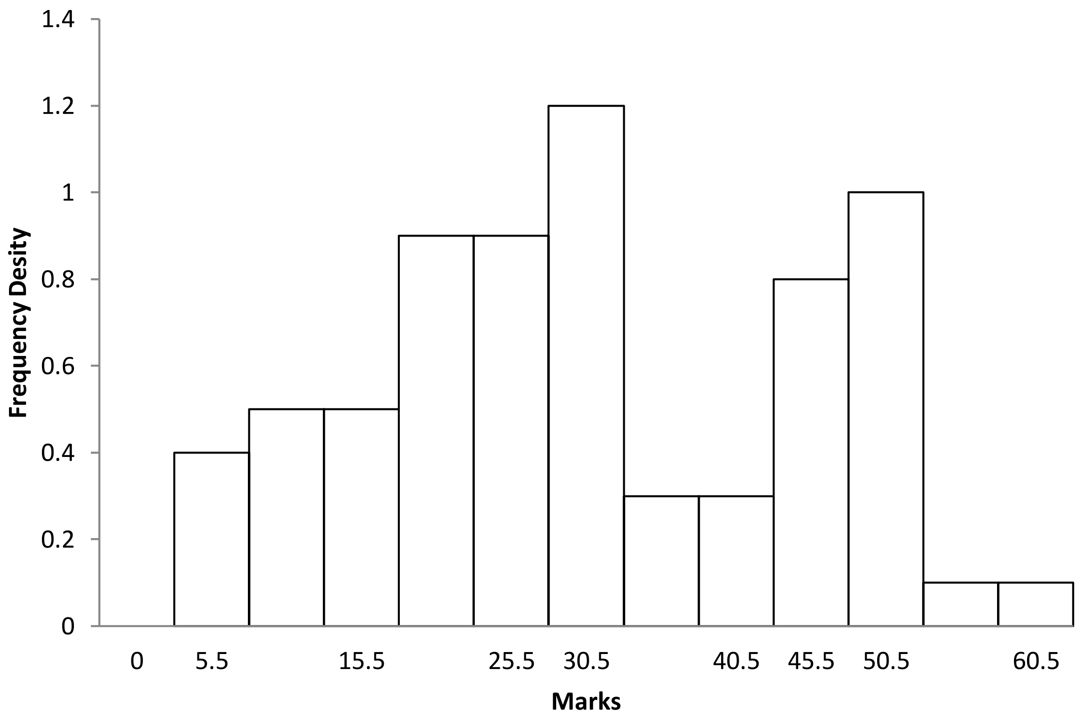
Frequency density is the ratio of the frequency and the size of the class in question.

Therefore, frequency density = $\frac{\text{frequency}}{\text{class size.}}$

Hope you know how to find the class size.

Draw the histogram of the following distribution.

No of Students	Frequency	Frequency Density	Class Midpoint	Class Boundaries	Class Size
1 – 5	2	0.4	3	0.5 – 5.5	5
6-15	5	0.5	8	5.5 – 15.5	10
15 – 25	9	0.9	15.5	15.5 – 25.5	10
25 – 30	6	1.2	25.5	25.5–30.5	5
31 – 40	3	0.3	35.5	30.5– 40.5	10
41 – 45	4	0.8	43	40.5– 45.5	5
46 – 50	5	1	48	45.5 – 50.5	5
51 – 60	1	0.1	55.5	50.5 – 60.5	10



Have you now seen how the bars are looking like? They are taking after the sizes of the class.

PIE CHART

This is the representation of data pictorially using the sectors of a circle. This is based on the idea of using the circle which has its total angle as 360 degrees and sharing that for the various items in question.

To draw a pie chart for a distribution, consider the following;

(i) Find the total category values

(ii) Calculate the angle of each category as if to share 360^o among the various items.

(iii) Use a compass to draw your circle and then protractor to do your demarcations for the various sectors.

Example,

The following data gives the monthly budget of a family at Madina.

Food – GH¢60

Clothing – GH¢10

House rent - GH¢50

Fuel and lightening - GH¢15

Miscellaneous - GH¢35

Savings - GH¢30

(a) Represent the information on a pie chart.

(b) What percentage of the monthly budget is house rent?

Solution

Total budget = 60+10+50+15+35+30=200

Angle of circle = 360^o

Therefore food = $\frac{60}{200} \times 360^o = 108^0$

Clothing = $\frac{10}{200} \times 360^o = 18^0$

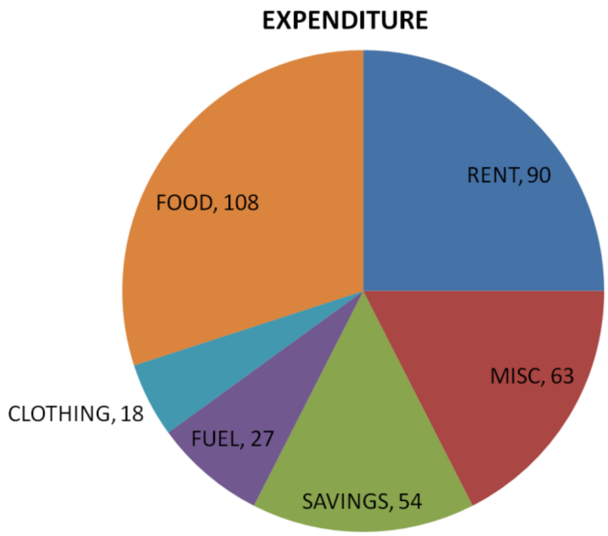
House rent = $\frac{50}{200} \times 360^o = 90^0$.

Fuel and lightening = $\frac{15}{200} \times 36^o = 27^0$.

Miscellaneous = $\frac{35}{200} \times 360^o = 63^0$

Savings = $\frac{30}{200} \times 360^o = 54^0$

After you have found the angles of the various sectors, you don't just draw your pie chart but rather add all the angles to see if they add up to 360 degrees.



(b) Percentage of house rent

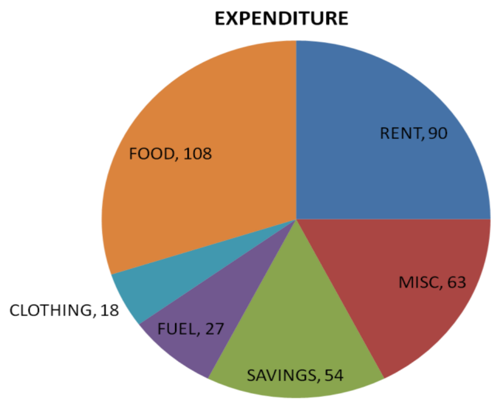
$$\frac{50}{200} \times 100 = 25\%$$

Interpretation of Pie Chart

Whenever a pie chart is already drawn for you, it is then required of you to interpret it or bring out the meaning by answering some questions that will be given.

For instance

The pie chart below shows how a student spent his pocket money of 180 Ghana Cedis. Use the pie chart to answer the following questions.



1. What fraction of the student's money did he spend on food?
2. How much does the student save supposing we did not know the angle for the savings?
3. How much is the total salary if he spends 200 Ghana cedis on food?

Solution

1. the angle on food is compared to the total angle of the circle hence,

$$\frac{108^o}{360^o} = 0.3.$$

2. The angle of the student's savings should be known first before knowing how much he saves.

Therefore let angle (savings) be x

$90^o + 108^o + 63^o + 27^o + x^o = 360$

$288^o + x = 360^o$

$X = 360^o - 288^o$

$X=72^0$

Now 72/360 x 666.67. Then the student's savings is Gh¢133.34

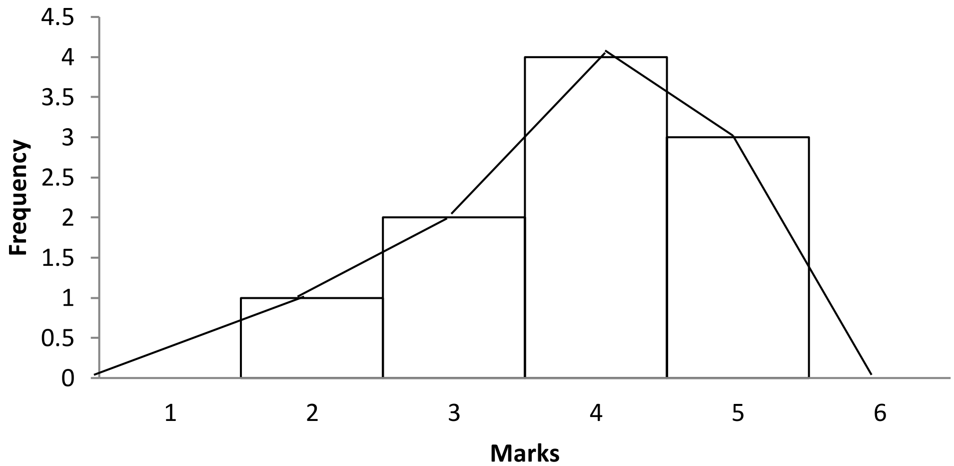
3. Let a variable stand for the total salary, hence, $\left(\frac{108}{3609}\right) \times A$ =200

Therefore, $\frac{(200 \times 360)}{180}$ = A

A = 666.67

Frequency polygon and Line graph

This is a line graph which may be drawn by joining the midpoints of the tops of the rectangles which will form a histogram. It is extended to the next lower and higher classes which are assumed to have zero frequencies to form a closed figure which is a polygon. The line graph takes almost the same trend like the polygon, only that it is not closed or extended to the next lower and higher classes.



Cumulative frequency curve.

A cumulative frequency distribution is needed to draw the curve. A cumulative frequency distribution table will be drawn for easy drawing of the curve.

The cumulative frequency for any class is the sum of the frequencies of that class and the lower classes.

The last frequency in the table should be equal to the total frequencies.

Study this

The following is the frequency distribution table of the marks scored by 120 candidates in an examination.

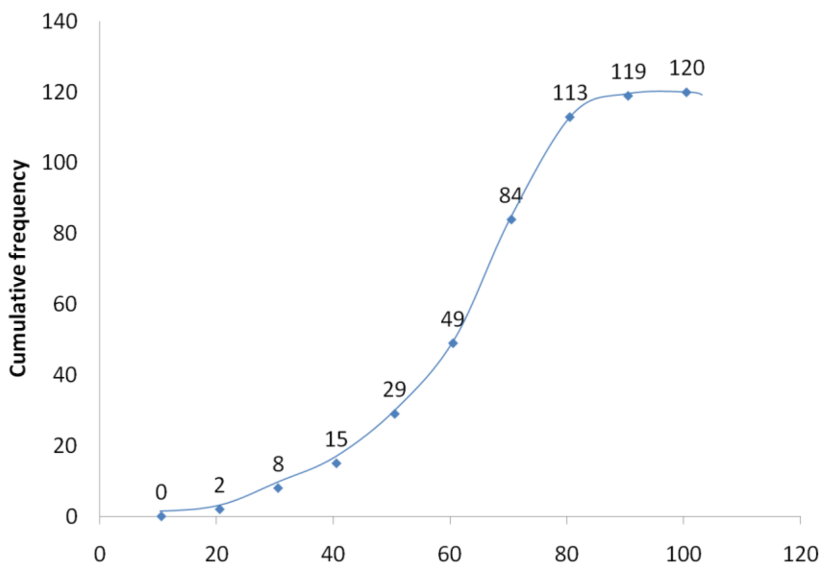
Marks	Frequency
1-10	0

11-20	2
21-30	6
31-40	7
41-50	14
51-60	20
61-70	35
71-80	29
81-90	6
91-100	1

Prepare a cumulative frequency distribution table and use it to draw a cumulative frequency curve.

Cumulative frequency table

Marks	Marks less than	Frequency	Cumulative freq
1-10	10.5	0	0
11-20	20.5	2	2
21-30	30.5	6	8
31-40	40.5	7	15
41-50	50.5	14	29
51-60	60.5	20	49
61-70	70.5	35	84
71-80	80.5	29	113
81-90	90.5	6	119
91-100	100.5	1	120



On the graph the upper class boundaries are always along the horizontal axis and the cumulative frequencies along the vertical axis.

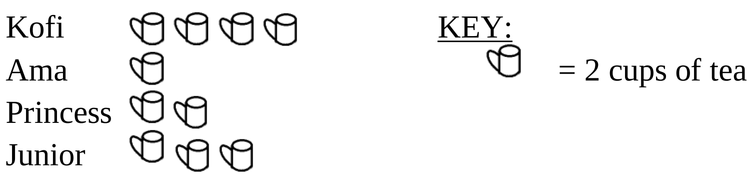
Joining all the consecutive points by a sharp curve is called cumulative frequency curve or Orgive.

THE PICTOGRAPH

The pictograph uses symbols to represent sizes of categories in data. The choose of the symbol is largely subjective , although many people tend to choose symbols that relate to the variable being represented. This si actually appropraite because, this pictograph is a qualitative representation of data and the use of it in the Basic Schools is very beneficial.

For example:

1. Supposing a cup represents two cups of tea for the individuals as shown below:

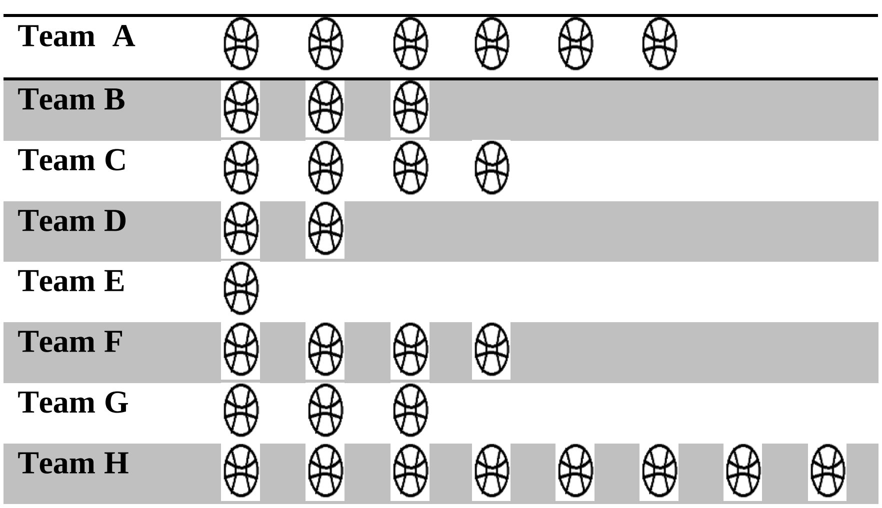


Then the following questions could be asked depending on the key.

- How many cups of tea can Kofi drink?
- How many less cups of tea can Princess drink than Kofi? etc.

The choice of symbol is largely subjective, although many people tend to choose symbols that relate to the variable being represented.

2. The pictograph shows the number of goals scored by eight teams who played in a recent football tournament.



KEY:

 = 1 goal scored

- How many goals were scored in total?
- Which team scored the least goal?
- Which team scored the highest goals

CENTRAL TENDENCY

It can be observed that most of the terms or values in a data seem to be crowded in its central part when data are arranged in increasing or decreasing order. A number around which there is concentration of terms of the data is called a measure of central tendency. Measures of central tendency are simply the averages of the frequency distribution. Central Tendency (CT) is a statistical *measure* that identifies *a single score* as a representative for an entire distribution or set of data.

Terms

• Measures of average are also called **measures of central tendency** and include the **mean**, **median**, **mode**, and **midrange.**

• Measures that determine the spread of data values are called **measures of variation or measures of dispersion** and include the **range, variance,** and **standard deviation**.

• **Measures of position** tell where a specific data value falls within the data set or its relative position in comparison with other data values. The most common measures of position are **percentiles, deciles,** and **quartiles.**

• The measures of central tendency, variation, and position are part of what is called **traditional statistics**. This type of data is typically used to confirm conjectures about the data.

• A **statistic** is a characteristic or measure obtained by using the data values from a sample.

The mode

The mode which can sometimes be called the modal value of a set of numbers is the number which occurs most frequently. It is the observations that possess maximum frequency in the given data. Mode or modal value is not the highest frequency but rather the number which gives the highest frequency.

Note that modal value can be one or more values in a data. The modal value may also not exist at all.

For instance when given

(a) 1, 1, 1, 2, 2, 1, 1, 2, 1, 2, 3, 3 the mode is 1; this is because it has the highest frequency of 6.

(b) 1, 1, 1, 2, 2, 1, 1, 1, 2, 2, 3, 2 and 2. The mode is 1 and 2. This is because both have the same frequencies and of course the highest of 6, 6. We say that, the distribution is bi- modal.

(c) 1, 2, 3, 4, 5, 6, 7, 8.

This seems to be counting numbers for which can continue till forever. This data has no modal value. The mode does not exist.

Mode of ungrouped frequency distribution

The mode is the value with the highest frequency.

With the ungrouped data it can be read directly from the frequency distribution table.

Example

Marks	Frequency
20	6
30	20
40	4
50	10

From the frequency distribution table, the highest frequency is 20; the mode of the data is 30. Measuring that, 20 students had the mark 30.

The mode of a grouped frequency distribution here, the modal class is the class with the largest frequency. For a grouped distribution, the whole class is taken into consideration.

Example

The table below shows the distribution at a certain train station, the number of minutes between the time a train stops and the time it takes off again was noted for 115 trains.

Time	Frequency
4 – 7	1
8 – 11	9
12 – 15	29
16 – 19	38
20 – 23	32
24 – 27	5

28 – 31	4

From the frequency distribution table, the modal class is 16 – 19

This is because it is 16 – 19 which has the highest frequency of 38.

The Mode

● The mode is the most frequent score in a distribution.

● It is the "typical" value.

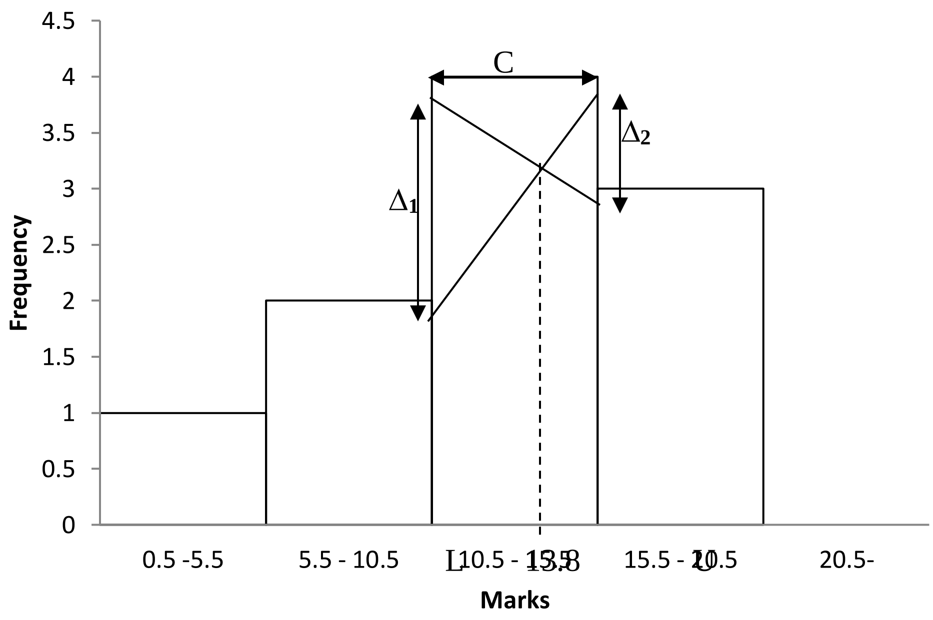
● In a frequency graph you can immediately see what the mode is because it is the tallest value, or the score with the highest frequency.

● Notation: M_o

The mode from a histogram.

The value of the mode within the modal class of a grouped frequency distribution can be found from a histogram of the distribution using the highest rectangle, relate it to the nearest ones to the left and right bars, find the point of intersection and then trace it mark to the mark for the modal value.

Illustration



The mode from a histogram by formula

The modal value can also be calculated from the modal boundary of the class, the width of the modal class and other lower class and the class values around it. In this case we use the formula

Mode = $L + \left(\frac{\Delta_1}{\Delta_1 + \Delta_2} \right) C$ where

L= Lower boundary of the modal class

Δ_1= modal class frequency – frequency next to lower class

Δ_2 = modal class frequency – frequency next to upper class

C = width of the modal class

Example

Calculate the mode of this distribution

Class	10-15	15-20	20-25	25-30	30-35	35-40
Frequency	2	12	27	41	30	7

Solution

By using the formula

Mode = $L + \left(\frac{\Delta_1}{\Delta_1 + \Delta_2} \right) C$

L = 25

Δ_1= 41- 27 = 14

Δ_2 = 41-30 = 5

Therefore, the mode = 25+$\left(\frac{14}{14+11} \right)$= 27.8

Mean

One of the popular averages of the frequency distribution is the mean. It is one of the numbers around which there is concentration of terms of the data. It is symbolized $\bar{X}$. We have the arithmetic mean, Geometric mean, harmonic mean, etc for our studies; we shall look at the arithmetic mean.

The mean of a set of numbers or values.

The arithmetic mean of set of numbers of say "n" values: x_1, x_2, x_3, $x_{4,}$.… x_n is given as

$\bar{x} = \left(\frac{x_1 + x_2 + x_3 + x_4 ... + x_n}{n} \right)$ since the numbers of terms are $\bar{x} = \frac{\sum x}{n}$ that is, sum of x divided by number of x.

The Mean

●The average of a set of scores

●The most commonly used measure of CT

● Notation is $\bar{x}$

●The mean of a population is symbolized as: m

●The mean of a sample is symbolized as: x (with a bar on top)

●The mean = the sum of all the scores divided by the number of scores: $\frac{\sum x}{N}$

Example

Find the mean of the following serves (1, 2, 5, 6, 5, 6, 2, 8, 10, 3)

$$\frac{\sum x}{n} = \frac{(1+2+5+6+5+6+2+8+10+3)}{10} = 4.2$$

The mean from a frequency distribution for ‘x’ values with respective frequencies, it means there are values like x_1, x_2, x_3, x_4 …. x_n with frequencies like f_1, f_2, f_3, … f_n. The mean therefore is given as $\bar{x} = \frac{\sum fx}{\sum f}$ When the frequency distribution is not grouped, then the individual marks multiplied by their respective frequencies, sum and then divide by the sum of the various frequencies.

Example,

No of babies	Mass of babies	fx
1	2	2
2	3	6
3	5	15
4	10	40
5	15	75
6	30	180
7	25	175
8	15	120
9	10	90

10	5	50
	Σf= 120	Σfx=753

$$\bar{x} = \frac{\sum fx}{\sum f} = \frac{753}{120} = 6.275$$

When the frequencies distribution is grouped, then we find the midpoints of the various classes and let them stand as x with the same formula.

Example,

Time	Frequency	Midpoint	fx
4-11	1	7.5	7.5
8-11	5	9.5	47.5
12-15	9	13.5	121.5
16-19	5	17.5	87.5
20-23	6	21.5	129
24-27	5	25.5	127.5
28-31	3	29.5	88.5
	Σf = 34	Σx= 124.5	Σfx= 609

$$\bar{x} = \frac{\sum fx}{\sum f} = \frac{609}{134} = 17.912.$$

When given the assumed mean.

When a data is given, repeated observation can be made. If we assume or guess that the mean is say A, we called that 'A' assumed mean.

To find the actual mean, with a grouped data the difference between the assumed mean and the class midpoint is found. It is called the deviation.

The mean is therefore found by the formula

$$\bar{x} = A + \frac{\sum fd}{\sum f}$$

Where $d = x\text{-}A$, x = class midpoint

Example, the following table shows the distribution of wages earned by some construction workers in Ghana. Find the mean wages when assumed mean = 64.5

Wages	40-49	50-59	60-69	70-79	80-89	90-99
No of workers	12	12	18	12	3	3

Solution

Wages	No of workers	Midpoint	Deviation (x-A)	fd
40-49	12	44.5	-20	-240
50-59	12	54.5	-10	-120
60-69	18	64.5	0	0
70-79	12	74.5	10	120
80-89	3	84.5	20	60
90-99	3	94.5	30	90
	Σf=60			Σfd=-90

Mean $(\bar{x}) = A + \frac{\sum fd}{\sum f}$ **,** $\bar{x} = 64.5 + \frac{90}{60}$

= 64.5+1.5 = 66

Median

The median is that value of the variable which divides the distribution into two equal frequencies. It is the middle value of a set of numbers or the arithmetic mean of the two middle values of a set of numbers when they are arranged in increasing or decreasing order.

To find the median for a set of discrete variables,

(a) arrange the numbers in order of magnitude (increasing or decreasing)
(b) if n is odd, then the median is the middle item, that is, the ½ (n+1) item.
(c) If n is even, then the median is the arithmetic mean of the two middle items ie the ½ (n^{th}) and $(½ n+1)^{th}$ item.

Median of a set of values

Example

Find the median of this set {3, 7, 4, 1, 2, 3, 6, 9, 8}

$\Rightarrow$ 1, 2, 3, 3, 4, 6, 7, 8, 9

Since the number (n) is odd, the median is the middle number = 4

In other words

N=9, median = $½(9+1)^{th}$ item = 10/2 = 5 , hence the 5^{th} item = 4.

Median from an ungrouped frequency distribution

Firstly, find the total frequency, that is, Σf then see whether it is odd or even if it is odd, then the median is given as ½ (Σf+1)th item, example, if Σf = 27, then, median = ½ (27 + 1) = 28/2 = 14th item.

If it is even; then the median is given as arithmetic mean of the two middle item.

Hence ½ (Σf) item.

Example if the Σf = 24

Then the median is the (12th) item and the item after it.

Example, find the median

Age	17	18	19	20	21
No of students	3	10	8	5	2

Solution

Find the sum of the frequencies = 28

Find half of *f*

28/2 = 14

Now 14th and 15th

Hence the median = 19.

To find the median of a frequency distribution, it is advisable to find the cumulative frequencies for easy working.

Median for a grouped distribution

The median of a grouped data is calculated the same way, only that we don't have one number but a class. Therefore we can talk of median class.

Marks	No of students
10-20	5
20-30	10
30-40	22
40-50	6
50-60	6
60-70	3

Solution

(i) Find the sum of the frequencies = 50

(ii) Find ½ (Σf) = 50/2 = 25th and 26th

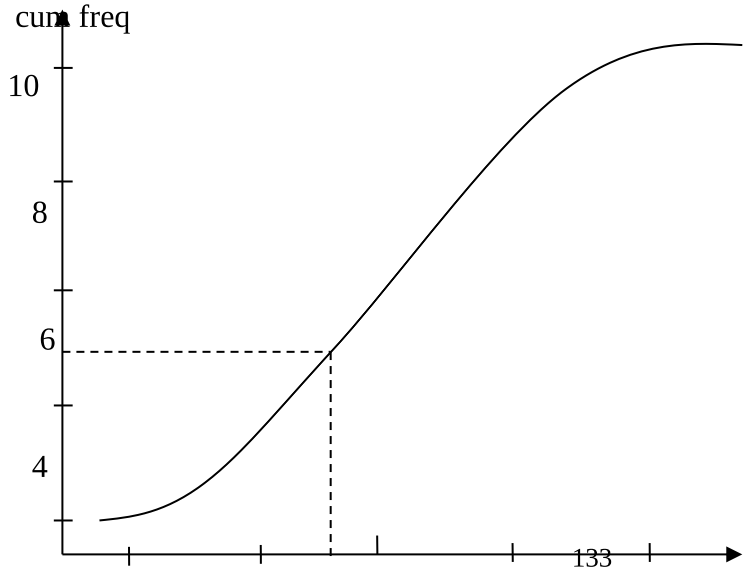
(iii) Now add one to get the 25th and 26th items. The 25th and 26th items fall under 20-40 class.

The median from a cumulative frequency curve.

It is used to estimate the median from a grouped data. From a cumulative frequency, the median is the value on the x-axis corresponding to half of the total frequency.

Study this

Marks less than	frequency	Cumulative frequency
5.5	2	2
10.5	1	3
15.5	3	6
20.5	2	8
25.5	2	10



2

0 5.5 10.5 15.5 20.5 25.5
Marks

The median can therefore be estimated to be about 14.1

The graph sheet will give you a very good estimate.

A formula can also be used to find the median.

Thus.

$$\text{Median} = L + \left(\frac{\frac{1}{2} N - \left(\sum f \right)_L}{f} \right) C$$

Where

L = lower boundary of median class

N = total frequency

$(\Sigma f)_L$ = sum of frequency below median class

f median = frequency of median class

C = with of the median class.

Measures of Central Tendency and Scales of measurement

· The mode requires only nominal data - and you can compute it for ordinal, interval, and ratio.

· The Median requires ordinal data - and you can compute it for interval and ratio. You cannot compute the Median for nominal data.

· The Mean requires interval or ratio data - you cannot compute it for either nominal or ordinal data.

Measures of central tendency and skewed distributions

· If the distribution is symmetrical and unimodal (like the normal distribution) the mean will equal the median which will equal the mode.

· If the distribution is skewed, these three measures of central tendency will not agree.

· A skewed distribution is a distribution that has a long tail extending out on one end. This is caused by having a few very extreme values relative to the majority of the scores.

· Positively and negatively skewed distributions

· Bimodal distributions

· Uniform distributions

Properties of the different central tendency measures:

● The mean is the standard measure of central tendency in statistics. It is most frequently used.

● The mean is not necessarily equal to any score in the data set

● The mean is the most stable measure from sample to sample.

● The mean is very influenced by Outliers - That is, the mean will be strongly influenced by the presence of extreme scores.

● The median is not sensitive to outliers.

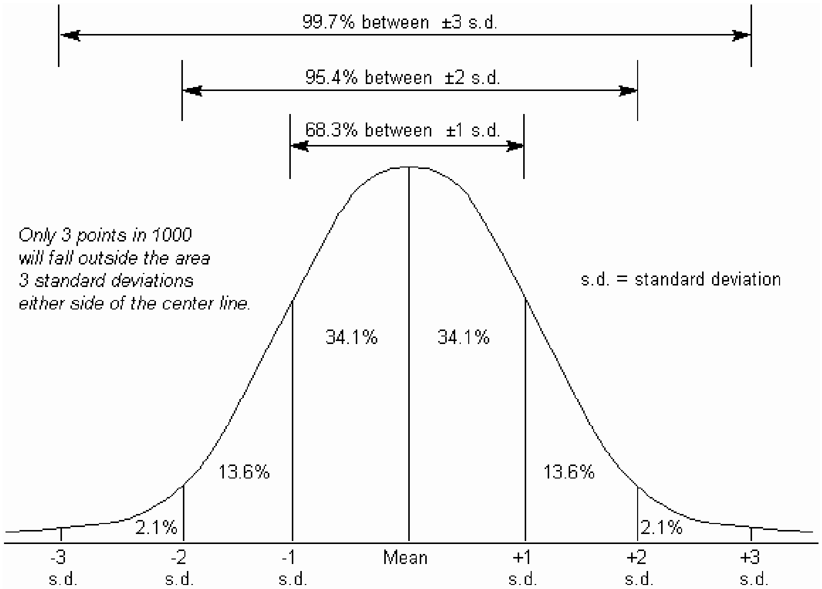
● The mean is based on all scores from the sample but the mode and the median are not.

● The Mode is the least stable measure from sample to sample.

● The median is the best measure of central tendency if the distribution is skewed.

Which central tendency measure you can use depends on the level of measurement your scores represent:

Level of Measurement	MODE	MEDIAN	MEAN
Nominal	Yes	No	No
Dichotomies	Yes	No	Yes
Ordinal	Yes	Yes	No
Interval/Ratio	Yes	Yes	Yes



Level of measurement

(From Wikipedia, the free encyclopaedia)

The level of measurement of a variable in mathematics and statistics is a classification that was proposed in order to describe the nature of information contained within numbers assigned to objects and, therefore, within the variable. The levels were proposed by Stanley Smith Stevens in his 1946 article on the theory of scales of measurement. Different mathematical operations on variables are possible, depending on the level at which a variable is measured. According to the classification scheme, in statistics the kinds of descriptive statistics and significance tests that are appropriate depend on the level of measurement of the variables concerned.

Four levels of measurement were proposed by Stevens:

* nominal,

* ordinal,

* interval and

* ratio.

Measures of skewness are concerned with whether the data are symmetrically distributed, or the shape of the distribution.

Most people are familiar with the distribution referred to as the normal, or bell-shaped, curve. Many of the statistics we use assume the data are distributed normally. Unfortunately, this is not always the case.

Symmetric distribution

In a distribution displaying perfect symmetry, the mean, the median, and the mode are all at the same point, as shown in Figure 1 .

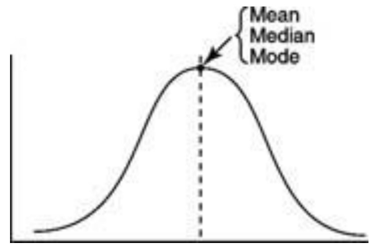


Figure 1 For a symmetric distribution, mean, median, and mode are equal.

Skewed curves

As you have seen, an outlier can significantly alter the mean of a series of numbers, whereas the median will remain at the centre of the series. In such a case, the resulting curve drawn from the values will appear to be **skewed**, tailing off rapidly to the left or right. In the case of negatively skewed or positively skewed curves, the median remains in the centre of these three measures.

Figure 2 shows a negatively skewed curve.

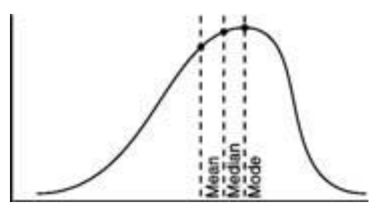


Figure 2 A negatively skewed distribution, mean < median < mode.

Figure 3 shows a positively skewed curve.

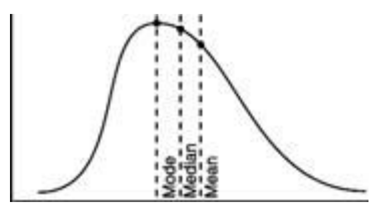


Figure 3 A positively skewed distribution, mode < median < mean.

Using measures of central tendency

The choice of any particular average depends on the type of data involved whether quantitative or qualitative. For quantitative data, it is possible to determine any of the three averages although there may be situations where one could be preferred to the other two. For example, the modal annual dues paid by clubs in a community are the best choice of average for a prospective member to have a fair idea to annual dues of clubs in that community. For qualitative data, the mean is not a reasonable value. As a matter of fact, if the qualitative data is nominal, then it is not also reasonable to find the median.

It is not possible to take the mean or the median as the average of data involving gender because the mean or the medium of data involving gender does not exist. Therefore, the average for such data is the mode. For example in a family of seven males and three females, where the mode is male, the average gender is male. Similarly, given shoe sizes 5,6,8,6,8,5,8,5, 11, 8, 10, 6, 5, 7, 8, 9, 7, 8, 6 and 7, the only reasonable average is the modal size i.e size 8, because the data set is ordinal, even though the categories involved are denoted by numbers. For the same reason as in the nominal case, the mean is not a possible average for data involving; say, ranks of soldiers. Consequently, the recommended averages are the median and the mode. For example, if a group of soldiers comprises 3 lieutenants, 2 captains, 2 majors, 1 colonel and 1 brigadier, then the modal rank is lieutenant and the median rank is captain**.**

Any of the three averages is a reasonable representation of continuous data and in some cases, of discrete data. For example, if the weight (in kg) of ten boxers is 65, 57, 63, 90, 71, 42, 63, 102, 89 and 66, then the mean, weight is 70.8kg. Another average for this data set is the median, 65.5kg. That is if the total weight of the ten boxers were divided equally among them, each boxer would weigh 70.8kg. Another average for this data set is the median, 65.5kg and the other half weighs above this value. The data set can also be represented by another average, the mode, which is 63kg. This scenario suggests that the three averages of a particular data set are not always equal.

For discrete data, note that the mean is not always a reasonable representation. For example, if the sizes of four classes 24, 32, 36 and 27. Then the mean is 29.75, which is an 'impossible' value for representing number of students.

The usage of the word average in everyday language does not always imply the mean. It may also imply the median or the mode. For example the statement: 'the average leave period for a worker is 30 days' is not to say that the mean of all leave periods is 30days. Therefore, the implied average here is the mode.

Some statements describing data do not explicitly mention the term mean, median, mode or even average, although any one of these measures may be implied. For example, the statement: 'Mr

Kert continued to re-examine his economics students until half of them scored 70% or more.''
This implies that, the median score is 70%.

MEASURES OF DISPERSION

The dispersion of a set of data is the amount of spread of the data. How the set of spread of the data is analyzed and measured is termed as the measure of dispersion.

Some of the measures of dispersion we can have are the range, inter-quartile range, semi inter-quartile range, standard deviation, mean deviation, variance and the sort.

By applying the measure of dispersion, we are able to assess the reliability of the average being used.

Range

The range of a set of data is the differences between the greatest or the largest and least or smallest item of the distribution.

Here we have

Largest item – smallest item.

The range from a set of numbers, for instance,

33, 35, 45, 11, 12

The range = largest – smallest item.

$\Rightarrow$ 45 – 11 = 34

Range = 34

From a frequency distribution,

For instance when given

X	1	2	3	4	5	6
F	1	3	9	2	1	1

Find the range

Solution

The range = largest item – smallest item

$\Rightarrow$ 6 – 1 = 5

The range = 5

Inter-quartile range

To find the inter-quartile range, you must know what the quartiles are.

Now,

The **quartiles** of a data divide the set of data into four equal parts.

The first one – fourth is called the **lower quartile** and denoted by **Q_1**. It is the ¼th of the total frequency hence $Q_1 = \frac{1}{4} \Sigma f$

When the data is divided into four equal parts, the third is called the **upper quartile.** It is denoted by **Q_3**.

The upper quartile is given by $Q_3 = \frac{3}{4} \Sigma f$

The second or the middle quartile is the **median,** which is denoted by **Q_2** . The median is given by $Q_2 = \frac{1}{2} \Sigma f$

We therefore have **the inter-quartile range** as the difference between the upper quartile and the lower quartile.

In this case if you want to find the inter-quartile range you need to find the quartiles first, before you manipulate or calculate to get the inter-quartile range.

The **inter-quartile** range is given as upper quartile (Q_3) – lower quartile (Q_1)

Example,

If a distribution has a lower quartile of 147 and an upper quartile of 166. Calculate the inter-quartile range

Solution

Inter-quartile range = Q_3-Q_1

$Q_3 = 166$ and $Q_1 = 147$

Therefore

Inter-quartile range = 166-147 = 19

Semi inter-quartile range.

The semi inter-quartile range is the **half** of the inter-quartile range. It is therefore given as

$\frac{1}{2}(Q_3 - Q_1)$

In this case, to calculate the semi inter-quartile range, you have to know the inter-quartile range.

Example

If a distribution has a lower quartile of 147 and upper quartile of 166. Calculate the semi inter-quartile range.

Solution

Semi quartile range is given as

$\frac{1}{2}(Q_3 - Q_1)$

$\frac{1}{2}(166 - 147)$

$\frac{1}{2}(19) = 9.5$

Note

The inter-quartile range and the semi inter-quartile range are slightly better measures of dispersion than just the range. This is because they are not affected by extreme values because they are based on the 'middle-half' of the data. The value of the quartiles calculated is then read from the values in the marks (x) column or from the cumulative curve not the value of the frequency. The same apply to the deciles and percentiles.

The quatiles could also be found by estimating them from the cumulative frequency curve just as was than for the median on the cumulative frequency curve. Hope you remember?

When a qrouped data is given, then apart from using the cumulative frequency curve to estimate, the formulae below could be used.

$$Q1 = L + \left[\frac{\left(\frac{n}{4} - fb\right)}{fq}\right] \times w$$

$$Q3 = L + \left[\frac{\left(\frac{3n}{4} - fb\right)}{\boxed{fq}}\right] \times W$$

Where,

L =lower class boundary of the interval that contains the quatile;

n=sum of the frequencies;

fb =cumulative frequency for all classes before the class containing the quartile;

fq = frequency of the class interval containing the quartile;

w =width of the class intervalcontaining the quartile;

Deciles and the percentiles

• **Measures of position** tell where a specific data value falls within the data set or its relative position in comparison with other data values.

• The most common measures of position are **percentiles, deciles,** and also the **quartiles**

As the name depicts, the **deciles** is the 10^{th} of the total frequency in the frequency distribution. In this case to find the deciles of a given distribution, it is the ratio one tenth of the total frequency. Example, if the total frequency of a distribution is given as 50, then the decile is found by (1/10)*50 = 5.Always remember that, the value calculated is just a position directing you to the distribution for the required **percentiles, deciles, quartiles** etc.

In the same way, the **percentiles** are the various percentages of the total frequency. We can therefore have the 25%, 50%, 60%, 75%, etc of the total frequency. For instance, when we are given the total frequency of a distribution to be 50, then the 60^{th} percentile is given as (60/100)*50 = 30, etc

Mean deviation

The mean deviation is simply a slight change found between the mean and the marks. It is usually measured from the arithmetic mean.

The mean deviation of a set of numbers; If we have a set of numbers as $x_1, x_2, x_3, x_4, \ldots x_n.$

When given the values 11, 12,13,14,15,

Find the mean deviation.

$$\bar{x} = \frac{\sum x}{n} = \frac{11+12+13+14+15}{5} = 13$$

$\sum\left|x-\bar{x}\right|$ = 11 – 13 + 12 – 13 + 13 – 13 + 14 – 13 + 15 - 13 = -2 + -1 + 0 + 1 + 2

= 6 by taking the absolute value.

Now $M.D = \frac{\sum|x-\bar{x}|}{n} \Rightarrow$ 6/5 = 1.2

B. The mean deviation from a frequency distribution table is given by

$$M.D = \frac{\sum f|x-\bar{x}|}{\sum f}$$

Where f = frequency, $\bar{x}$ = the mean and x the marks.

For a grouped frequency distribution the x is the midpoint values found.

Example,

Find the mean deviation from the distribution below

Class	10-15	15-20	20-25	25-30	30-35	35-40
Frequency	2	12	27	41	30	7

In a simple way, create a table, that is,

Class	Frequency	Midpoint	fx	$\lvert x-\bar{x}\rvert$	$f\lvert x-\bar{x}\rvert$
10-15	2	12.5	25	14.45	28.90
15-20	12	17.5	210	9.45	113.40
20-25	27	22.5	607.5	4.45	120.15
25-30	41	27.5	1127.5	0.55	22.55
30-35	30	32.5	975	5.55	166.50
35-40	7	37.5	262.5	10.55	73.85
	Σf=119		Σfx =32075		$\sum f\lvert x-\bar{x}\rvert = 525.35$

$$\bar{x} = \frac{\sum fx}{\sum f}$$

Mean deviation, $M.D = \frac{\sum f|x-\bar{x}|}{\sum f} = \frac{525.35}{119}$

= 4.41(2 d.p)

All calculations must be done systematically to avoid mistakes.

Variance and Standard Deviation.

The variance and the standard deviation are measures of dispersion which move together all the time.

To calculate the standard deviation of a data, it will be that, the variance might have been worked out first before the standard deviation. This is because; the Standard deviation is the positive square root of the variance. In that case if you have a variance say 9, the standard deviation is √9 which is equal to 3.

1. The variance and standard deviation of a set of numbers. When a set of numbers like x_1, x_2, x_3, x_4,… x_n, is given, the variance is given as

Variance = $\frac{\sum\left(x-\bar{x}\right)^2}{n}$

Whilst the standard deviation as

$$S = \sqrt{\frac{\sum\left(x-\bar{x}\right)^2}{n}}$$

Where $\bar{x} = \frac{\sum x}{n}$, that is, the simple mean.

n= the number of items. x = the individual items.

$$V = \frac{\left(\sum x\right)^2}{n} - \left(\bar{x}\right)^2$$

$$\Rightarrow S = \sqrt{\frac{\sum x^2}{n} - \left(\bar{x}\right)^2}$$

2. Variance and standard deviation from ungrouped data; variance is given as;

Variance = $\frac{\sum fx^2}{\sum f} - \bar{x}^2$

And standard deviation (s) as;

$$\Rightarrow S = \sqrt{\frac{\sum fx^2}{\sum f} - \left(\bar{x}\right)^2}$$

Where f = frequency and $\bar{x} = \frac{\sum fx}{\sum f}$

The same formula is used for grouped data where the x is taken from the midpoints of the classes.

Example

Find the variance and the standard deviation of 11, 12, 13, 14 and 15.

Finding the mean gives 11+12+13+14+15/ 5 = 65/ 5 = 13

Now after getting the mean from this simple data, it will be easier to form a table. Hence

X	x- $\bar{x}$	$(x-\bar{x})^2$
11	11-13=-2	4
12	12-13=-1	1
13	13-13=0	0
14	14-13=1	1
15	15-13=2	4
		$\sum(x-\bar{x})=10$

From the table $\sum(x-\bar{x})=10$

Variance = $\frac{\sum(x-\bar{x})^2}{n}$ = 10/5 = 2

For the standard deviation

$S = \sqrt{\frac{\sum(x-\bar{x})^2}{n}}$ = √5 = 1.414

Example 2

Find the standard deviation of the data below

X	*F*	*F*x	fx^2	x^2
1	2	2	2	1
2	3	6	12	4
3	5	15	45	9
4	10	40	160	16
5	15	75	375	25
6	30	180	1080	36
7	25	175	1225	49
8	15	120	960	64
9	10	90	810	81
10	5	50	500	100
	Σf=120	Σfx= 753	Σfx^2=5169	

$$\frac{\sum fx}{\sum f} = \bar{x} = \frac{753}{120} = 6.273$$

$$V = \frac{\sum fx^2}{\sum f} - \left(\bar{x}\right)^2 = \frac{5169}{120} - (6.275)^2$$

$= 43.075 – 39.375625$

$= 3.699375$

$$\Rightarrow S = \sqrt{\frac{\sum fx^2}{\sum f} - \left(\bar{x}\right)^2}$$

$$= \sqrt{\frac{5169}{120} - (6.275)^2} = \sqrt{3.699375}$$

S= 1.92

Example 3

Find the variance and the standard deviation of the data below.

Class	*F*	Midpoint	x^2	*f*x	fx^2
1-10	8	5.5	30.25	44	242
11-20	14	15.5	240.25	217	3367
21-30	28	25.5	650.25	714	18207
31-40	42	35.5	1260.25	1491	52930.5
41-50	35	45.5	2070.25	1592.5	72458.75
51-60	16	55.5	3080.25	888	49284
	Σf=120			Σ*f*x= 4446.5	Σfx^2=196489.25

$$\bar{x} = \frac{\sum fx}{\sum f} = \frac{4446.5}{143} = 31.09$$

$$\text{Variance} = \frac{\sum fx^2}{\sum f} - \left(\bar{x}\right)^2$$

$$= \frac{196489.25}{143} - (31.09)^2$$

$= 1374.05 – 966.5887$

V= 407.46

Standard deviation =

$$S = \sqrt{\frac{\sum fx^2}{\sum f} - \left(\bar{x}\right)^2} = \sqrt{407.46}$$

S= 20.19

Measures of position and variability tell us where the data are located and how dispersed they are.

Measures of skewness are concerned with whether the data are symmetrically distributed, or the shape of the distribution.

Most people are familiar with the distribution referred to as the normal, or bell-shaped, curve. Many of the statistics we use assume the data are distributed normally. Unfortunately, this is not always the case.

STRAND 8 BASIC PROBABILITY:

LEARNING, TEACHING AND APPLYING

THE CONCEPT OF PROBABILITY

Life is full of uncertainties and as a human being, there are a whole lot of decisions you will have to make to assume a successful life. Despite trying very hard to take decisions that will make your life successful, some of the decision may shutter your life forever. Sometimes you may even not know the full consequences of your actions. A helpful evaluation of our decisions, being it at home, workplace, and wherever you will find yourself is the concept of probability**.** Probability is the study of events that may or may not occur. It is a study of randomness of uncertainty. It is just like a game of chance. The likelihood of something occurring or not occurring is associated with probability.

Probability concepts allow us to make more reliable forecasts and predictions, even when we have only limited information. Probability concepts are very important tools in decision making, in statistical analysis and in many research areas.

You might have asked some of the following questions in one time or the other

(a) What is the chance of Ghana Black-Stars in beating Brazil in this "finals" of the World cup?

(b) How sure are you that the next elections will be free, fair and transparent?

(c) Has Princess any chance of becoming the next beauty queen in the Miss Ghana context?

(d) How sure are you that it will rain this evening?

(e) will I get a head or a tail when my fifty pesewas coin is thrown?

(f) When I throw a die right now, what face will appear? A six, five, four, three, two or one?

(g) what is the chance of me throwing my orange in the air for it to come back?

All these questions and thousands of them bring to fore the concept of Probability.

From the questions above, it could clearly be seen that not all the outcomes could be predicted so easily and be sure of it occurring, whilst others are predictable and very sure of it occurrence. For instance, when I throw my orange in the air, I can predict it will come down and I am very sure it will. But when I throw a die, as to which side will show up will have to wait until it lands, though I may prefer a six.

To study the concept of probability so that it will be useful for decision making and the sort, then the following terms should be looked at.

1. Experiment

Experiment is any action or process that generates observations. With this, the outcome cannot be predicted with certainty. E.g. the tossing of a coin.

2. Trial

A trial of an experiment is a single performance of the experiment.

E.g. tossing a coin once

3. Sample space

The sample space of an experiment denoted by S is the set of all possible outcomes of the experiment e.g. When a coin is tossed once the possibility outcomes are head/tail

4. Random sampling

Random sampling is choosing a sample from a population without being biased.

5. Population is the set of object or items under study. E.g. human beings

6. Sample

Sample is part of population e.g. women out of human beings

7. Sample point

Each element in the sample space is defined as a sample point. E.g. When a coin is tossed, head and tail are sample points.

8. Event

Event denoted by A is a collection of sample points with a common property. It is a subset of sample space $\Rightarrow$ A C S

For events we have,

i. Equally likely events

These are events which have equal chances of occurring. In literal terms, we will say 50/50 chance.

ii. Compound events

Events can be combined by the words 'or' and 'and'. Events which are thus combined are called compound events. The words 'or' and 'and' correspond to union and intersection with respect to sets. That is,

AUB denotes A or B (or both)

A∩B denotes A and B, where both events occur together. In probability, ‘or’ means addition and ‘and’ means multiplication.

iii. Mutually exclusive events

Two events are said to be mutually exclusive if they cannot occur together. For instance, if A and B are events such that (A∩B) = Ø, then we say that they are mutually exclusive

Example when you toss a coin once you will get a head or a tail. You cannot get a head and a tail at the same time. We say getting a head and a tail are mutually exclusive.

iv. Independent events

Two events say A and B are said to be independent if the occurrence of A does not depend or affect the occurrence of B and vice versa.

When two coins are tossed so that the first toss you get is a head and the second toss you get is another head, the event A which gave a head had no influence or effect on the second toss or event B. they are independent event.

Note the following

i. Probabilities are numbers between 0 and 1 inclusive. They can also be expressed as percentages between 0% and 100%.

ii. Probabilities near 0 indicates that the event in question is not likely to occur

iii. Probabilities near 1 indicates that the event in question is likely to occur

iv. Probabilities near ½ indicates that the event in question is has about the same chance of occurring as it has of failing to occur.

POPULATION AND SAMPLING

Population is the total number of subjects (which are not necessarily people) of your research that conform to a clearly defined set of characteristics. You are always collecting data on characteristics (variable) that varies within the population, and you are assuming that there is a spread of values across this population. The theoretical spread of this quality across the population is called the normal distribution of the variables concerned.

If a population is drawn and the results generalized to the population as a whole to ensure confidence in the reliability to our conclusions, we use probability sampling to select the sample to make sure it is representative of the population as a whole. The subjects of your study are collectively referred to as the population.

Sampling a Population

In an ideal world, you will collect a research data from the entire population. This is only possible if the population is small. In the case where the population is very large, then you will not have the time and resources to collect the data of the whole population.

The only way around this is to select a population of the total population. A collection of a proportion of the total population is what is termed as sample. Logic suggests that, if conclusions are to be generalized to the whole population, the samples should be representative of the population in some ways. The techniques used to select the sample are collectively called Probability Sampling.

CLASSICAL APPROACH TO PROBABILITY

After studying the concept of probability and knowing almost all the terms, the classical approach to probability; is assumed that, the events of an experiment are equally likely and mutually exclusive. Events are said to be **equally likely** if each event has the same chance of occurring, for instance, when a coin is thrown, both the head and the tail are equally likely to occur. Events are said to be **mutually exclusive** if only one of the possible events of an experiment can occur at a time, for instance when a die is thrown only one side can occur at a time. The classical approach to probability was developed in the 17^{th} century and was widely used in games of chance.

The probability of event can be computed by dividing the number of outcomes in the experiment by the total number of possible outcomes of the experiment.

The relationship can be expressed as:

Probability of an event = $\frac{\text{number of outcomes in the event}}{\text{total number of possible outcomes}}$

Hence P (A) = $\frac{\text{number of ways event A can occur}}{\text{total number of possible outcomes}}$

Note the following

i. Probabilities are numbers between 0 and 1 inclusive. They can also be expressed as percentages between 0% and 100%.

ii. Probabilities near 0 indicates that the event in question is not likely to occur

iii. Probabilities near 1 indicates that the event in question is likely to occur

iv. Probabilities near ½ indicates that the event in question is has about the same chance of occurring as it has of failing to occur.

If the probability that an event A will occur denoted by P (A) = 'r' then the probability that the event will not occur is denoted by P(A) = 1 – P(A) = 1 – r

Example if the probability that Ghana wins is ¾ then the probability that Ghana will not win is ¼

The probability that a universal set is certain to occur is 1, that is, P(U) = 1.

P (Ø) = 0 where Ø is null set.

Probabilities of event occurring or not occurring always lie between 0 and 1as already stated.

Now;

Suppose an event 'A' can happen in M ways out of a total of n ways, then the probability of occurrence of event A is given as P (A) = $\frac{m}{n}$

Hence P (A) = $\frac{\text{number of ways event A can occur}}{\text{total number of possible outcomes}}$

1. All possible outcomes of an experiment are denoted by n(S)
2. The number of ways in which the event A could occur is denoted by n(A) then, the probability that event A will occur is given by P(A) = $\frac{n(A)}{n(S)}$

P (A) is used as the possibility of event A.

Examples

(i) A box contains 10 beads for which 6 are red and 4 are white. If a bead is chosen without looking, what is the probability that a white bead is chosen?

Solution

n(S) 10
n(red) =6
n(White) = 4

P (White) = $\frac{n(\text{ white})}{n\ (s)}$ = 4/10 = 2/5

(ii) In a class of 40 students, 15 are girls and 25 are boys. If one student is to be chosen for a scholarship package randomly, what is the probability that a girl will be chosen?

Solution

P(g) = $\frac{n(g)}{n(S)}$

= 15/40 = 3/8

(iv) What is the probability of drawing an ace from a shuffled deck of 52 playing cards?

Solution

With this, you must first of all know how many ace are there in 52 playing cards.

n(S) = 52
n(A) = 4
P(A) = $\frac{4}{52} = \frac{1}{13}$

(v) A fair ludo die is tossed once.
(a) What is the probability of observing 3?
(b) What is the probability of observing an odd number?
(c) What is the probability of observing any number ?

(a) The event has only one outcome, which is 3.

Total number of possible outcomes is is 6, since the die has six sides.

The die is fair ,so the possible outcomes are equally likely.

Therefore,

$$\text{Probability of 3} = \frac{\text{number of outcomes in the event}}{\text{total number of possible outcomes}}$$

$$\frac{1}{6}$$

(b) The odd numbers are 1 3 and 5.

Therefore the event is made up of three outcomes

Total number of possible outcomes is is 6, since the die has six sides.

The die is fair ,so the possible outcomes are equally likely.

Therefore,

Probability of observing an odd number

$$= \frac{\text{number of outcomes in the event}}{\text{total number of possible outcomes}} = \frac{3}{6} = \frac{1}{2}.$$

Relative frequency

Probabilities can be estimated from the results or outcomes of an experiment

$$\text{The experiment probability} = \frac{\text{number of times the outcome happens}}{\text{to number of times the experiment occurs.}}.$$

This is sometimes called the ***relative frequency.***

Consider the following frequency distribution table for the ages of students in a class.

Ages	21	22	23	24	25	26	27	28	29
Frequency	2	1	1	2	3	2	5	3	1

Find the probability that a student selected at random from the class is

(i) 24 years old
(ii) Less than 24 years
(iii) Greater than 26 years
(iv) Between 23 and 27 years.

Solutions

Total frequency = 20

(i) Let A be the event of being 24 years

n(A) = 2

P (A) = n(A) /n(S) = 2/20 = 1/10

(ii)Let B be the event of being less than 24 years

n(B) ie Less than 24years = 2+1+1=4

P(B) = 4/20 = 1/5

(iii)Let C be the event of being older than 26 years

n(C), ie more than 26 = 5+3+1 =9

P(C) =9/20

(iv)Let D be the event of being between 23and27.

n(D) ie between23 and 27 = 2+3+2

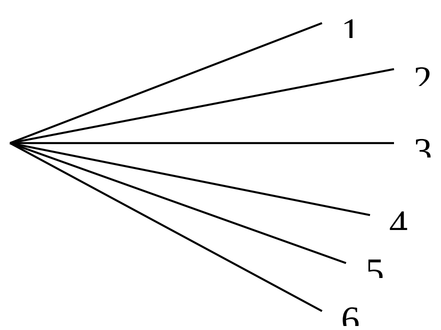
P(D) 7/20

Probability tree diagram

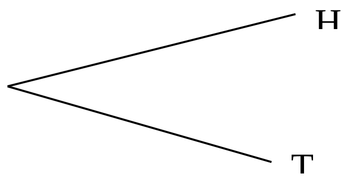
A probability tree diagram can be used to solve problems involving compound or combined events.

Using the tree diagram just look like having numbers which of course are probabilities on the branches

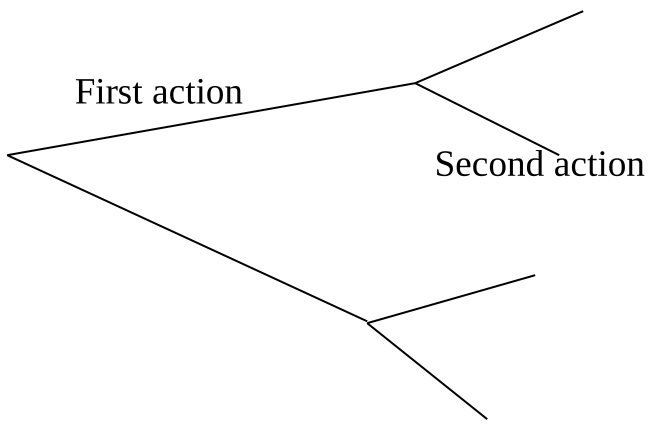
E.g. Tossing a die will give



Or a coin as



If the event has a second time, i.e. of the compound event we may have two stops with the necessary branches. E.g.



At the end of each route along the branches of the tree, you write its final outcome on the branches, workout the probability of each of the outcome and write on its branch of the tree.

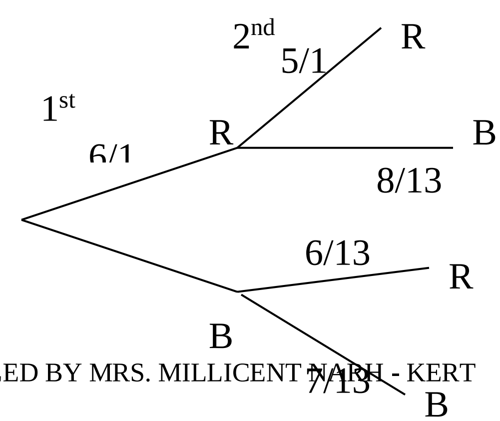
Make sure that probabilities on adjacent branches of the tree add up to one (1)

- To find the probability of the final outcome, find the route along the branches which lead to the outcome. You multiply using the multiplication law.
- For example, a bag contains 6 red and 8 blue balls; two of the balls are drawn at random one after the other without replacement. (a) Draw a tree diagram to illustrate the outcomes of the experiment. (b) Display on the diagram the probability of each of branch of the tree

Solution

Let R be red balls then n(R) = 6 B be blue balls, then n(B) = 8

Total number of balls = 6+8 = 14



(c) what is the probability that

(i) A blue ball was drawn and then red

(ii) both are of the same colour.

(iii) both are of different colour .

Solution

(i) P(Band R) = 8/14 x 6/13 = 48/182 = 24/91

(ii) P(both of the same colour) P(R and R) or P(B and B) = 6/14x 5/13 + 8/14x 7/13= 48/182+ 48/182= 96/182=43/91

(iii) P(both balls of diff colour) P(RB) or P(BR) = 6/14x 8/13 + 8/14x 6/13= 48/182+ 48/182= 96/182=43/91

Now that we know the nitty-gritty of probability lets solve some few questions on probability of simple event, relative frequency, mutually exclusive events, independent event, tree diagrams and conditional probability.

Probability of simple event

1. In a class of 12 boys and 18 girls what is the probability that a student chosen from the class is a boy/

   Solution

   n(B) = n(B) = 12 n(G) = 18 n(U) = 30 P(B) = n(B) /n(U)= 12/30= 2/5

2. A bag contains 5 red beads and 10 white beads. What is the probability that a bead chosen at random is white b

   Solution

n(red beads) = 5 n(white beads) = 10 , n(Universal set) 5+10= 15

then P(W) = n(W) / n(U) = 10/15 = 2/3

Probability of relative frequency

1. Three Ghanaian coins were tossed 50 times and the number of crest appearing on each time was recorded as follows

Number of crest	0	1	2	3
Frequency	13	8	12	17

Use the relative frequency to estimate the probability of two crest showing when three coins are tossed.

2. The table below shows the ages of 25 students in a certain Basic School.

Age	6	7	8	9	10	11	12	13
Frequency	4	3	1	3	6	2	4	2

If a student is chosen at random, find the probability that the age will be 11 years or more.

Solution.

P(age being 11 years or more) 2+4+6/25 = 8/25

Probability of mutually exclusive event

1. A fair coin is tossed twice, one after the other. What is the probability that first toss is a head and the second is also a head?

Solution

Sample space

n(U) =(HH,HT,TH,TT)

First toss is ½ and the second toss ½ first and second = ½ x ½ = ¼

Or from the sample space, two heads is 1 hence

P(exactly two heads) =n(E)/(U) =1/4

2. A box contains 14 white balls and 6 black balls. Find the probability that a black ball was drawn at first, replaced and then a second ball drawn is white

Solution

P(B) = n(B) /n(U) = 6/20= 3/10 .

P(W) = n(W) / n(U) = 14/20 = 7/10. Hence 3/10 x 7/10 = 21/100

Probability of independent event

Two fair dice, A and B each with faces numbered 1 to 6 are thrown together.

(a) Construct a table showing all the equally likely outcomes.

(b) From the table list the pair of numbers on the two dice for which the sum is (i) 5 (ii) 10 (iii) more than 10 (iv) at least 10

(c) Find the probability that two of the dice show (i) different scores (ii) same scores

(d) Find the probability that the sum of the numbers on the two dice is (i) 5 (ii) 10(iii) more than 10 (iv) at least 10

Solution

The table below shows the possible outcomes. The first column represent die A and the first row represent die B. The total number of possible outcomes n(U) =36.

B \ A	1	2	3	4	5	6
1	1,1	1,2	1,3	1,4	1,5	1,6
2	2,1	2,2	2,3	2,4	2,5	2,6
3	3,1	3,2	3,3	3,4	3,5	3,6
4	4,1	4,2	4,3	4,4	4,5	4,6
5	5,1	5,2	5,3	5,4	5,5	5,6
6	6,1	6,2	6,3	6,4	6,5	6,6

From the table, the pair of numbers whose sum is

(i) 5 are (1,4) (2,3) (3,2) and (4,1)

(ii) 10 are (4,6) (5,5) and (6,4)

(iii) More than 10 are (5,6) (6,5) and (6,6)

(iv) At least 10 are (4,6) (5,5) (6,4) (5,6) (6,5) (6,6)

(c) The Total number of equally outcomes = 36

(i) P(dice shows different scores) = $\frac{30}{36} = \frac{5}{6}$

(ii) P(dice show the same scores) = $\frac{6}{36} = \frac{1}{6}$

(d)Total of equally likely outcomes = 36.

The probability that the sum of the numbers on the two dice is

(i) 5 is $\frac{4}{36} = \frac{1}{9}$

(ii) 10 is $\frac{3}{36} = \frac{1}{12}$

(iii) More than 10 is $\frac{3}{36} = \frac{1}{12}$

(iv) At least 10 is $\frac{6}{36} = \frac{1}{6}$

Probability of conditional event

1. A box contains 10 red marbles and 3 white marbles. Three marbles are drawn at random one after the other without replacement. Find the probability that

   (i) All of the three are of the same colour.

   (ii) One of them is of different colour.

Solution

Let R = red marbles, n(R) = 10

W = White marbles n(W) = 3

Total = 3+10 =13

(i) N(three of the same colour) = P (RRR)n or P(WWW)

$$\Rightarrow \frac{10}{13} \times \frac{9}{12} \times \frac{8}{11} + \frac{3}{13} \times \frac{2}{12} \times \frac{1}{11}$$

$$= \frac{720}{1716} + \frac{6}{1716} = \frac{726}{1716}$$

$$= \frac{363}{858} = 0.4231$$

(ii) One of different colour P(RRW) or P(RWR) or P(WRR) or P(WWR) or P(WRW)or (RWW)

$$= \left(\frac{10}{13} \times \frac{9}{12} \times \frac{3}{11}\right) + \left(\frac{10}{13} \times \frac{3}{12} \times \frac{9}{11}\right) + \left(\frac{3}{13} \times \frac{10}{12} \times \frac{9}{11}\right) +$$

$$\left(\frac{3}{13} \times \frac{2}{12} \times \frac{10}{11}\right) + \left(\frac{3}{13} \times \frac{10}{12} \times \frac{2}{11}\right) + \left(\frac{10}{13} \times \frac{3}{12} \times \frac{2}{11}\right)$$

$$= \frac{270}{1716} + \frac{270}{1716} + \frac{270}{1716} + \frac{60}{1716} + \frac{60}{1716} + \frac{60}{1716}$$

$$= \frac{990}{1716} = 0.5769 = 0.58$$

(iv) P(both of the same colour) P(R and R) or P(B and B) = 6/14x 5/13 + 8/14x 7/13= 48/182+ 48/182= 96/182=43/91

(v) P(both balls of diff colour) P(RB) or P(BR) = 6/14x 8/13 + 8/14x 6/13= 48/182+ 48/182= 96/182=43/91

On Sundays, Kotoko and Hearts play football match at the Accra sports stadium. The probability that Kotoko scores a goal is 0.8 and the probability that hearts scores is 0.9. Calculate the probability that

(a) They both score
(b) Kotoko scores but hearts does not.
(c) None of them score
(d) At least one of them scores
(e) Use a tree diagram to illustrate the probabilities

Solution

(a) Both score
= 0.8 x 0.9 = 0.72

(b) Kotoko scores but hearts does not;
Kotoko = 0.8 and hearts is 0.1=

= 0.8 x 0.1 = 0.08

(c) None of them scores =
0.2 x 0.1 = 0.02

(d) At least one scores =

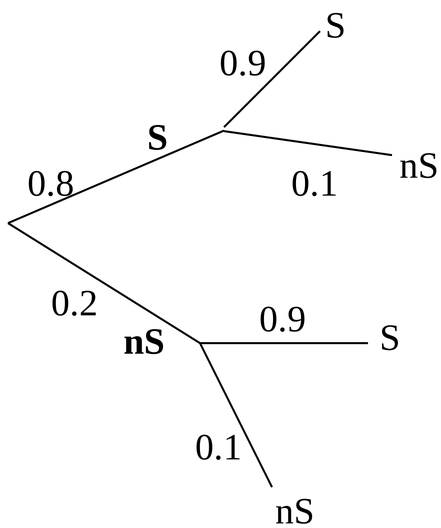
1 – P(neither scores)

= 1 - 0.02

= 0.98

(e) If score = S

And no score = nS



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